THEMATIC WORKING GROUP 4
ARGUMENTATION AND PROOF

Maria Alessandra Mariotti
Univeristà di Siena
Kirsti Hemmi
Stockholm University
Viviane Durrand Guerrier
Université de Lyon

The contributions collected in this section differently address the issue of proof and argumentation, offering a quite varied spectrum of perspectives, from both the point of view of theoretical frameworks assumed and that of issues in focus.
The richness of contributions' diversity gave the participants the opportunity of a fruitful discussion far beyond the need of sharing a common terminology, while the reflection, carried out at the beginning of our working activity highlighted the problematic relationship between argumentation and mathematical proof from the diversity of our theoretical and cultural backgrounds. The papers presented and discussed during the working sessions at CERME5 are collected in this chapter, organized according two different main themes.

1. Models and theoretical constructs to investigate argumentation and proof

The first group of papers exemplifies how different theoretical constructs may contribute to shape investigations, directing the researcher both in selecting the questions to be addressed and the ways to look for possible answers.
The role of epistemological analysis emerges in the paper of Castagnola & Tortora, where the famous Euclid's theorem on the infinity of prime numbers is presented as a paradigmatic example to discuss students' difficulties and to propose possible means for their overcome. The epistemological issue is also clearly addressed in the paper of Deloustal-Jorrand, where mathematical implication is analysed from three different points of view: formal logic point of view, deductive reasoning point of view, sets point of view. A *didactical engineering*, based on the assumption of the necessity of make these points of view interact is carefully described and its implementation discussed, showing how a suitable situation can raise the issue of implication. A complementary example is presented in Gibel's paper, where, in the frame of the *Theory of Didactic Situations (TDS)*, the author discusses the inadequacy of the situation that fails to engage students in solving a problem and consequently does not make it possible for the teacher to bring the students to valid mathematical reasoning.
Besides approaches consistent with general theoretical models, such as that of the TDS, specific theoretical constructs were presented, elaborated for the particular aim of analysing issues related to argumentation and proof. This is the case of the papers presented by Pedemonte and by Antonini & Mariotti. The construct of Cognitive Unity combined with the Toulmin model provides Pedemonte a powerful tool to

analyse and discuss the complex relationship between argumentation and proof in the special case of producing a conjecture. Antonini & Mariotti discuss the complexity of indirect proof using an interpretative model set up as refinement of the notion of Theorem introduced by Mariotti (200?).

Styliadis & Styliadis bring to the attention of mathematics education researchers a rich body of psychological research on deductive reasoning, related to the well known paradigm of mental models (Johnson-Laird, P.N., 1983). Such a theoretical construct may offer new insight not only in identifying important issues that require research attention, but also suggesting an interdisciplinary and collaborative approach to the problem of promoting proof in students' learning of mathematics.

Still a different perspective and a new theoretical model is drawn from cognitive psychology (Kahnman, 2002) and chosen by Buchbinder & Zaslavsky. Their objective is that of describing the students' behaviour when they are asked to decide truth-value of a statement; the model of the dual process theory is applied in order to describe (and explain) the differences between modes of justifying the truth-value of a statement, according to the notion of "confidence" (respectively "lack of confidence") that the subject has in its truth or its falsity.

A theoretical model is set up purposefully by Timmermann to describe the relationship that an individual (either the teacher or the student) can establish with a proof. Distinguishing between structure, components and details textbook proofs are analysed so as their presentation in class. A possible failure in communication between the teacher and the students may explain difficulties, and can be described in terms of discrepancies between what each interlocutor has in focus: a component or a detail.

As is the case for investigations based on the development of a teaching experiment, a twofold objective characterizes the study presented by Fiallo & Gutierrez: on the one hand to design a didactic intervention, on the other hand to study students' performances. An unusual mathematical domain is selected, trigonometry, and a teaching sequence is set up with the intention of introducing students to proof; in order to show its potentialities a specific model is elaborated for the analysis students' learning achievements.

A more explorative approach is taken by Aylon & Even. Differently from the previous studies, characterized by being structured and directed by the theoretical frame selected, this study aims to examine and classify opinions about the role played by learning mathematics in the development of general deductive reasoning. The interviewees are persons involved in mathematics education and logic. In spite of an expected variability, the findings of this study show a certain convergence on considering the development of general deductive reasoning as a goal of mathematics instruction, and on the assumption that to some degree this goal is attainable.

2. Teachers' beliefs and teacher practice

The realty of the classroom, as emerged from some of the previous contributions, brings to the forefront the centrality of the teacher's point of view, both in terms of

beliefs and in terms of the practices that in such beliefs find their origins.
This perspective is explicitly taken in the contribution presented by Sergis, illustrating how teacher's view of what constitutes a proof and its functions influences the choice of what is to be integrated into one's own teaching practices. He proposed an interesting issue concerning the relationship between teachers' beliefs and practices and the presentation showed the high complexity of treating this issue and the need of elaborating specific methodologies.
Shifting the attention on teachers' practices and aiming to explain how successfully a teacher teaches proof, Ding and Jones propose to analyse teaching of geometrical proof, as it is developed in Shangai classrooms. The analysis is carried out by using the van Hiele model and indicates that though the second and third van Hiele teaching phases could be identified in the Chinese lessons, the instructional complexity does not allow the full sequence of phases to be recognized.
If a global description of a successful practice seems far from being achieved, a more focused contribution comes from Gibel's paper. The author highlights the crucial role of the situation for a student to engage him/herself in an argumentative reasoning, nevertheless the episode discussed in the presentation raised the issue of the role of the teacher in supporting the student in overcoming the difficulty.
Teachers' beliefs and teachers' practices become a particular issue when the mathematics class involves prospective teachers. A long term teaching experiment is presented in the paper of Camargo, Samper & Perry, aimed to introduce future math teachers to deductive geometry. Among a number of different aspects characterizing this teaching experiment, the authors discuss on the mediating role that a Dynamic Geometry Environment, Cabri, can play.
The case of trainee teachers is still in focus in the paper presented by Cusi & Malara. Beyond the general difficulties related to proof, the analysis of proof production in elementary number theory show the specificity of this mathematical context.
As far as the task is assumed to require the coordination of verbal and algebraic registers, the authors show how the difficulties encountered in the solution processes can be explained by the lack of coordination.

3. Final conclusions

After the short presentations of the papers and the following debate, the working group participants broke into small groups. The discussion in the groups was guided by a set of issues related to the theoretical and methodological elements raised by the debate.

1. Using formal models for investigating proof and comparison between different models for investigating proof.
2. Proof in the classroom: focus on the task, focus on the mathematical domains focus on the teacher.
3. Teaching experiments for investigating proof: methodological issues related to investigating proof in the school context.

It is difficult to report conclusions for a group that spend considerable time facing the complexity of diversity. What emerged, rather than points of unanimous agreement, is the convergence on the centrality of some issues.

In our discussions it became clear that there is a need to specify the meaning of the terminology we are using, in particular in relation to the aim of dealing with analogies and differences between argumentation and proof; but equally importantly there is a need to understand better their inter-relations and relevance in the context of mathematics and of mathematics education as well.

The variety and the richness of the different theoretical frameworks utilized in the different contributions stimulated not only the need of comparison, but also the curiosity of undertaking a possible integration and this, in our view, constitutes the main result of our working sessions in the spirit of CERME.

INDIRECT PROOF: AN INTERPRETING MODEL[♣]

Samuele Antonini
Department of Mathematics
University of Pavia, Italy

Maria Alessandra Mariotti
Department of Mathematics
University of Siena, Italy

Starting from a general discussion on mathematical proof, a structural analysis was carried out, leading to the construction of a model within which indirect proofs can be described. The model shows itself a good interpreting tool to identify and explain cognitive and didactic issues, as well to precisely formulate research hypotheses concerning students' difficulties with indirect proofs.

INTRODUCTION

Mathematics education literature offers a rich and varied panorama of theoretical frameworks within which different didactic issues related to proof were identified and studied (see for instance Balacheff, 1987; Duval, 1992-93; Harel et al., 1998; Garuti et. al.,1998; Mariotti et. al., 1997; Pedemonte, 2002). However, the research studies carried out within these frameworks only rarely addressed issues that can be related to some specific logical structure of a proof. Some authors focused their attention on proof by mathematical induction (Harel, 2001; Pedemonte, 2002), not much attention was devoted to indirect proofs, with which, on the contrary, this paper intend to deal.

Although not always easily comparable, research studies devoted to indirect proof report consistent results concerning students' difficulties with this type of proof, that, at any school level, seem to be greater than those related to direct proof. Different interpretations and different sources of these difficulties were proposed. Certainly, as some authors remarked, indirect proofs do not find an adequate attention in school practice, at any school level (Thompson, 1996; Bernardi, 2002). But, although correct, this remark cannot fully explain the difficulties that students seems to face. Taking a cognitive perspective, other studies contributed to identify specific aspects that, affecting students' cognitive processes, could be responsible of their difficulties. For instance, assuming and dealing with false hypotheses was recognized as one of the main sources of difficulties (Leron, 1985). As far as the production of a proof is concerned, Wu Yu et al. (2003) and Antonini (2001, 2003a) demonstrated how difficulties could be found at the very beginning, when there is the need of correctly formulating the negation of statement. The distinction between different functions of a proof led to interpret students' difficulties (Barbin, 1988), to analyse conjecture generation processes that can lead students to produce a proof by contradiction (Antonini, 2003b), to propose

[♣] This research study was supported by the Italian Ministry of University and Research - Prin 2005 # 2005019721.

innovative didactic approaches to indirect proofs (Polya, 1967, pp. 163-171). Similarly differences were described in relation to how verbal and symbolic context affect students' performances when dealing with contraposition equivalence (Stylianides et al., 2004). The aim of this paper is outlining a frame allowing a uniform and systematic approach overcoming the fragmentary of previous contributions. What we present is an interpreting model of indirect proof, as well some examples aimed to illustrate its effectiveness.

METHODOLOGY

Our project developed through two main research directions, empirical and theoretical; the dialectical relationship between them contributed to the construction of the interpreting model. Empirical data were both qualitative and quantitative and were collected according to different methodologies: individual interviews, written tests, observation and recording of classroom activities, mainly collective discussion. Empirical investigations concerned both high school and university students (in particular students of scientific faculties: Mathematics, Physics, Biology, Pharmacy).

THE MODEL

The elaboration of the model started from the analysis of indirect proof that we framed within the model introduced in (Mariotti et al., 1997; Mariotti, 2000) through the ‘didactic’ definition of ‘mathematical theorem’. According to such a definition, a mathematical theorem is characterized by the system of relations between a statement, its proof, and the theory within which the proof makes sense. In the following the tern constituted by Statement, Proof and Theory, will be referred as (S, P, T).

The refinement of this definition was elaborated with the objective of taking into account two basic aspects: the logical structure of an indirect proof and the distinction between Theory and Meta-Theory.

Direct and indirect Proof

First of all, let us try to clarify what we mean with the expression *indirect proof*. In fact, the use of the expressions "indirect proof", "proof by contradiction", "proof by contraposition", "proof ad absurdum" in the textbooks is far from being clear and uniform, while its use may be considered controversial even among the mathematicians (Bernardi, 2002; Antonini, 2003a). In general, these expressions denote one or both of the types of proof which, in this paper, we call *proof by contradiction* and *proof by contraposition*. From logical point of view, there are important differences but also several relationships between them that we do not treat here; in the teaching of mathematics, in Italy, both of them are generally named “*dimostrazione per assurdo*” while sometimes this expression is used only for proof by contradiction (Antonini, 2003a).

In this paper, by *indirect proof* we refer to both *proof by contradiction* and *proof by contraposition*. In other words, consider a statement S and assume that it can be formulated as an implication, $p \rightarrow q$; let us name *direct* the proof P if among the statements that constitute the deductive chain does not appear the negation of the thesis p. If this is not the case, we will speak of *indirect proof*. Elaborating on this basic definition we are going to describe the structure of indirect proofs. Consider the following examples.

Example 1
Statement: Let n be a natural number. If n^2 is even then n is even.
Proof: Assume n a natural odd number, then there exists a natural number k such that n=2k+1. As a consequence $n^2 = (2k+1)^2 = 4k^2+4k+1 = 2\bullet(2k^2+2k)+1$, that means that n^2 is an odd number.

This is an example of "proof by contraposition". If the statement is expressed by the implication $p \rightarrow q$, the given proof is a **direct proof** of the new statement "*if n is odd then n^2 is odd*", that is the contraposition ($\neg q \rightarrow \neg p$) of the original statement $p \rightarrow q$.

Example 2
Statement: Let a and b be two real numbers. If ab=0 then a=0 or b=0.
Proof: Assume by contradiction that ab=0 and that $a \neq 0$ and $b \neq 0$. Since $a \neq 0$ and $b \neq 0$ one can divide both sides of the equality ab=0 by a and by b, obtaining 1=0.

This is an example of a "proof by contradiction", where is given a **direct proof** of the statement "*let a and b be two real numbers; if ab=0 and $a \neq 0$ and $b \neq 0$ then 1=0* ". The hypothesis of this new statement is the negation of the original statement and the thesis is a false proposition ("1=0").

In both examples, in order to prove a statement S, that we will call the *principal statement*, a direct proof is given of a new statement S*, that we will call the *secondary statement*.

Principal statement S	Secondary statement S*
Let n be a natural number. If n^2 is even then n is even.	Let n be a natural number. If n is odd then n^2 is odd.
Let a and b be two real numbers. If ab=0 then a=0 or b=0.	Let a and b be two real numbers. If ab=0 and $a \neq 0$ and $b \neq 0$ then 1=0.

Table 1
Principal statement and secondary statement in two indirect proofs

The two examples of proof share a common feature in the shift from one statement (principal statement) to another (secondary statement) and base the acceptability of this shift on particular logical relationships valid in the meta-theory.

In both cases, it is possible to prove that an indirect proof of the principal statement can be considered accomplished if the *meta-statement* S*→S is valid; in fact, in this case, from S* and S*→S it is possible to derive the validity of S by the well known "modus ponens" inference rule. But, the validity of the implication S*→S depends on the logic theory, i.e. the *meta-theory*, within which the assumed inference rules are stated. As it is commonly the case, i.e. in the *classic* logic theory, such a *meta-theorem* is valid, but it does not happen in other logic theories, such as the *minimal* or *the intuitionistic* logic[1].

Of course, no trace of this meta-theoretical elaboration is made explicit in the proofs as they are usually presented both the textbooks and in the courses.

A model of indirect proof

According to the previous analysis three key statements and three key theorems are involved in an indirect proof. The statements are: the principal statement S, the secondary statement S* and the implication S*→S, that we can name *meta-statement*, referring to its meta-theoretical status. The theorems are:

1. the *sub-theorem* (S*,C,T) consisting of the *statement S** and *direct proof* C based on a specific mathematical theory T (Algebra, Euclidean Geometry, and the like);
2. a *meta-theorem* (MS, MP, MT), consisting of a *meta-statement* MS=S*→S and a *meta-proof* MP based on a specific *Meta-Theory,* MT (that usually coincides with classic logic);
3. the *principal theorem*, consisting of the *statement* S, the *indirect proof of* S, based on a theoretical system consisting of both the *theory* T and the *meta-theory* MT.

Let name an *indirect proof of S a pair consisting in the sub-theorem (S*,C,T) and the meta-theorem (MS,MP,MT); in symbols P=[(S*,C,T),(MS,MP,MT)].* In summary, an indirect proof consists of a couple of theorems belonging to two different logical levels, the level of the mathematical theory and the level of the logic theory.

Statements	Proofs	Theoretical levels
S*	C direct	Theory T
S*→S	MP	Meta-Theory (MT)
S	(S*,C,T)+(MS,MP,MT) indirect	T+MT Theory and Meta-Theory

Table 2

Analysis of an indirect proof. We highlighted the only elements usually made explicit in a mathematical proof as we read it in a textbook.

[1] For a definition in terms of rules of inference of the classic, minimal and intuitionistic logic, see Prawitz (1971).

HYPOTHESES EMERGING FROM THE INTERPRETING MODEL

The model presented above emerged from an a priori structural analysis but can be reinvested to describe and analyse students' cognitive processes, involved both in producing and interpreting indirect proofs. In particular, this model allows to identify key elements to formulate research hypotheses concerning the potential source of students' difficulties. In the following we will discuss some of them.

1) The shift from the proof of the principal statement to the proof of the secondary statement may present specific difficulties; in fact, the relationship between the two proofs may not be so *intuitively acceptable* (in the sense of Fischbein, 1987) as is commonly assumed.
2) Cognitive conflict may be expected to emerge in either producing or interpreting the proof of the secondary statement, when inferences are made on the base of openly false hypotheses.

We are going to illustrate these hypotheses through the discussion of two examples; we hope also to illustrate the effectiveness of the model in framing the analysis. In the transcript of the interviews the letter "I" indicates the interviewer while the initial of the name indicates the student. Emphasis is expressed by bold characters.

Difficulties in the shift from the principal to the secondary statement

Fabio is a university student (last year of the degree in Physics), he was asked to express his opinion on the indirect proof [2].

PROTOCOL	ANALYSIS
F: Proof by contradiction is artificial: how does one get out of it? Ok, you have arrived to the contradiction… and then? […] I don't see that conclusion be linked to the other one, I miss the spark […]	Fabio clearly express his difficulty to grasp the link between the two statements S and S*: according to our model the source of difficulty seems to be the meta-theorem.
I: Let's make an example: we take a natural number n. Theorem: if n^2 is even then n is even. Proof: if n is odd I write n=2k+1, then... [the interviewer writes down algebraic transformations] $n^2=2(2k^2+2k)+1$ is odd.	The interviewer proposes an example. The principal statement is: S : *if n^2 is even then n is even*. The indirect proof consists of the direct proof of the secondary statement, S*, that remains

[2] Fabio and the interviewer use the expression "proof by contradiction" to denote "proof by contraposition". As previously reported, many students and teachers in Italy, name in this way both proof by contradiction and proof by contraposition, without distinguish between the two types.

	unspoken: *if n is odd then n^2 is odd.*
F: Yes, I understand, it is better to prove that if n is odd then n^2 is odd.	Fabio acknowledges the advantage of the shift to the secondary statement S*, that he makes explicit.
I: And then, which is the problem?	
F: The problem is that in this way we proved that n is odd implies n^2 is odd, and I accept this; but I do not feel satisfied with the other one.	Fabio clearly express his feeling. He can identify the two statements, he is ready to accept the given proof, C, as a proof of S* ("*we proved that n is odd implies n^2 is odd, and I accept this*") but not as a proof of S ("*I do not feel satisfied with the other one*").
I: Do you agree that natural numbers are odd or even and there are not other possibilities?	
F: Yes, of course... and now you will say: n^2 is even, n is even or odd, but if it were odd, n^2 would be odd, but it was even... yes, ok, I know, but…something escapes me.	Fabio shows that is able to produce an argument to explain the method of the indirect proof, nevertheless there is something that he is not able to grasp ("*something escapes me*").
F: First of all, why do I have to begin from n not even? I don't see immediate conclusion. And, at the end: "then it can not be other than n even", it is a gap, the gap of the conclusion... it's an act of faith... yes, at the end it's an act of faith.	The move from the proof of the secondary statement to the validation of principal statement is not immediate, on the contrary is not rationally acceptable.

As the analysis of the protocol shows, the acceptability of the proof of the statement S* does not immediately entail that the principal statement was proved. The feeling of distress (upset) openly expressed by Fabio was also observed in other studies. For instance Stylianides et al. (2004) observed a similar resistence:

"*some students reject the contraposition equivalence rule because they believe that the correct equivalence relating the conditional statement $p \Rightarrow q$ with the proposition $\neg p$ and $\neg q$ is $p \Rightarrow q \equiv \neg p \Rightarrow \neg q$*" (p. 149).

What makes this protocol so peculiar is the fact that the introspection ability of Fabio let us know where the conflict arises. According to our hypothesis and with the terminology of our model, the difficulty can be localized in the meta-theorem, that is in the cognitive difficulty of grasping as immediate the logical link expressed by the implication S*→S. For Fabio, and probably for many other students, such a link is not an intuition (in the sense of Fischbein, 1987; for other details on intuition and the shift between statements, see also Antonini, 2004) and accepting it is cause of distress ("*I don't see immediate conclusion*", "*I do not feel satisfied*", "*something escapes me* ").

Difficulties in identifying the Theory of reference

The following example concerns specific difficulties related to producing a proof of a secondary statement, when the inferences should be based on openly false assumptions. Maria is a university student of the last year of Pharmacy. (For other details on the proof of the secondary statement, see Antonini & Mariotti, 2006).

PROTOCOL	ANALYSIS
I: I: Could you try to prove by contradiction the following: "if ab=0 then a=0 or b=0"?	The interviewer asks Maria a proof by contradiction.
M: [...] well, assume that ab=0 with a different from 0 and b different from 0... I can divide by b... ab/b=0/b... that is a=0. I do not know whether this is a proof, because **there might be many things that I haven't seen**.	Maria produces such a proof, but she is doubtful about its validity.
M: Moreover, so as ab=0 with a different from 0 and b different from 0, that is against my common beliefs [in Italian: *"contro le mie normali vedute"*] and I must pretend to be true, **I do not know if I can consider that 0/b=0. I mean, I do not know what is true and what I pretend it is true**.	Difficulties emerge about the validity of the sub-theorem: upsetting fundamental beliefs seems to be the cause. Maria declares that she lost the control on what is true and what is false.
I: Let us say that one can use that 0/b=0.	
M: It comes that a=0 and consequently … we are back to reality. Then it is proved because … also in the absurd world it may come a true thing: thus I cannot stay in the absurd world. **The absurd world has its own rules, which are absurd,** and if one does not respect them, comes back.	Maria considers "*absurd*" the "*world*" where the false hypothesis of the secondary statement is assumed. The "*rules*" (Theory) in respect to which the proof of the sub-theorem makes sense, belong to an "*absurd world*"; these rules are absurd too, they may not coincide with the rules commonly applied.
I: Who does come back?	
M: It is as if a, b and ab move from the real world to the absurd world, but the rules do not function on them, consequently they	Where we accept something false, Maria claims, whatever can happen, included 0/b≠0. The absurdity of the hypothesis of the secondary statement conflicts with use of the 'common' Theory; Maria thinks that T should be

have to come back …

M: But my problem is to understand which are the rules in the absurd world, are they the rules of the absurd world or those of the real world? **This is the reason why I have problems to know if 0/b=0, I do not know whether it is true in the absurd world**. […]

replaced by a new theory T*, different and more adequate to the *absurd world* where the proof is deplaced. The *real world* has its own rules, different from those of the *absurd world* .

I: [The interviewer shows the proof by contradiction of "$\sqrt{2}$ is irrational", than asks] what do you think about it?

M: in this case, I have no doubts, but why is it so? … perhaps, when I have accepted that the square root of 2 is a fraction I continued to stay in **my world**, I made the calculations **as I usually do**, I did not put myself problems like "in this world, a prime number is no more a prime number" or "a number is no more represented by the product of prime numbers". The difference between this case and the case of the zero-product is in the fact that this is obvious whilst I can believe that the square root of 2 is a fraction, **I can believe that it is true and I can go on as if it were true**. In the case of the zero-product I cannot pretend that it is true, **I cannot tell myself such a lie and believe it too!**

Differently from the previous case, Maria declares that she is comfortable with this proof; in fact, assuming the $\sqrt{2}$ is rational does not present any difficulty for her: the fact that $\sqrt{2}$ is rational is plausible. Consequently, in the *absurd world,* where $\sqrt{2}$ is rational, the basic truths are not questioned ("*I can believe that it is true and I can go on as if it were true*"), the Theory of reference is not upset.

The source of Maria's difficulties can be found in the difficulty of managing the upset of well settled rules caused by an assumption that is so evidently false for Maria that she cannot even pretend that it is true without upsetting the whole theoretical frame within which any proof can make sense. According to our model, Maria difficulties concern the sub-theorem (S*,C,T) and in particular the identification of theory to which the proof C refers. Maria hypothesises that a new theory T* should be used, different but more adequate to the absurd situation generated by the false assumption, in which she does not know whether 0/b is equal or not to 0.

This result agrees with what expressed by Durand-Guerrier (2003): implication with false antecedent are not accepted or however are considered as false by students. In order to accept a proof it seems necessary to start from true, or at least potentially true, assumptions.

CONCLUSIONS

Although the brevity of this text does not allow further discussion, we hope that the previous examples gave an idea of how the interpretative model may function in the analysis of students difficulties. In particular the model allows to clarify the articulation of the different theorems involved (*principal theorem, sub-theorem* and *meta-theorem*), and the articulation of the different theoretical levels (*theory* and *meta-theory*). A generic difficulty related to indirect proof can be consequently analysed in a more refined way, focussing on different aspects, showing the appearance of quite different kind of 'difficulties'.

An open research question is how to design teaching interventions aimed either to foster students' introduction to indirect proofs or to help students to overcome cognitive conflicts. We think that the model could show its effectiveness for planning didactic interventions, but in this regards, further investigation is necessary.

References

Antonini,S.(2001), Negation in mathematics: obstacles emerging from an exploratory study, *Proceedings of the 25th PME Conference, Utrecht, The Netherlands*, v. 2 pp. 49-56.

Antonini,S.(2003a), *Dimostrare per assurdo: analisi cognitiva in una prospettiva didattica*, Tesi di Dottorato, Dipartimento di Matematica, Università di Pisa.

Antonini,S.(2003b), Non-examples and proof by contradiction, *Proceedings of the 2003 Joint Meeting of PME and PMENA*, Honolulu, Hawai'i, U.S.A., v. 2 pp. 49-55.

Antonini,S.(2004), A statement, the contrapositive and the inverse: intuition and argumentation, *Proceedings of the 28th PME*, Bergen, Norway, vol. 2, pp. 47-54.

Antonini,S. & Mariotti, M.A. (2006), Reasoning in an absurd world: difficulties with proof by contradiction, *Proceedings of the 30th PME Conference, Prague, Czech Republic,* v.2 pp. 65-72.

Balacheff,N.(1987), Processus de preuve et situations de validation, *Educational Studies in Mathematics*, v. 18 n.2 pp. 147-176.

Barbin,E.(1988), La démonstration mathématique: significations épistémologiques et questions didactiques, *Bulletin APMEP* 366, 591-620.

Bernardi,C.(2002), Ricerche in Didattica della Matematica e in Matematiche Elementari, *Bollettino Unione Matematica Italiana*, Serie VIII, v. V-A pp. 193-213.

Durand-Guerrier,V. (2003), Which notion of implication is the right one ? From logical considerations to a didactic perspective, in *Ed. Studies in mathematics, 53,*5-34.

Duval,R.(1992-93), Argumenter, demontrer, expliquer: coninuité ou rupture cognitive?, *Petite x* 31, pp.37-61.

Fischbein,E.(1987), *Intuition in science and mathematics*, Dordrecht: Kluwer.

Garuti,R.,Boero,P.,Lemut,E.(1998), Cognitive Unity of Theorems and Difficulties of Proof, in *Proceedings of the 22th PME Conference, Stellenbosch, South Africa,* v. 2 pp. 345-352.

Harel, G., (2001), The development of mathematical induction as a proof scheme: A model for DNR-based instruction. In Campbell, S. & Zazkis, R. (Eds.), Learning and teaching number theory. *Journal of Mathematical Behavior* (special issue).

Harel,G.& Sowder,L.(1998), Students' Proof Schemes: results from exploratory studies, in A. Schoenfeld , J. Kaput and E. Dubinsky (eds.), *Research on Collegiate Mathematics Education*, v.3, M.M.A. and A.M.S. pp. 234-283.

Leron,U.(1985), A Direct approach to indirect proofs, *Educational Studies in Mathematics* v. 16 (3) pp. 321-325.

Mariotti,M.A.(2000), Introduction to proof: the mediation of a dynamic software environment, *Educational Studies in Mathematics* v. 44 pp.25-53.

Mariotti,M.A., Bartolini Bussi,M., Boero,P., Ferri,F., Garuti,R.(1997), Approaching geometry theorems in contexts: from history and epistemology to cognition, *Proceedings of the 21th PME Conference, Lathi, Finland* v.1 pp. 180-195.

Pedemonte,B.(2002), *Etude didactique et cognitive des rapports de l'argumentation et de la démonstration dans l'apprentissage des mathématiques*, Thèse, Université Joseph Fourier, Grenoble.

Polya,G.(1967), *How to solve it*, Princeton University Press, 1945. Traduzione italiana, *Come risolvere i problemi*, ed. Feltrinelli.

Prawitz,D.(1971), Ideas and results in proof theory. In J.E. Fenstad (editor), *Proceedings of the second Scandinavian Logic Symposium*, Amsterdam, pp. 235-307.

Reid,D.&Dobbin,J.(1998), Why is proof by contradiction difficult?, *Proceedings of the 22th PME Conference, Stellenbosch, South Africa* v. 4, pp. 41-48.

Stylianides, A.J., Stylianides, G.J., & Philippou, G.N. (2004), Undergraduate students' understanding of the contraposition equivalence rule in symbolic and verbal contexts, in *Educational Studies in Mathematics* v. 55 pp. 133-162.

Thompson,D.R.(1996), Learning and Teaching Indirect Proof, *The Mathematics Teacher* v. 89(6) pp. 474-82.

Wu Yu,J.,Lin,F.,Lee,Y.(2003), Students' understanding of proof by contradiction, *Proceedings of the 2003 Joint Meeting of PME and PMENA, Honolulu, Hawai'i, U.S.A.,* v. 4, pp. 443-449.

MATHEMATICS LEARNING AND THE DEVELOPMENT OF GENERAL DEDUCTIVE REASONING

Michal Ayalon Ruhama Even

Weizmann Institute of Science

This study aims to examine the approaches of people involved in mathematics education and logic to the role played by learning mathematics in the development of general deductive reasoning. The data source includes 21 individual semi-structured interviews. Analysis based on the Grounded Theory method identified three distinct groups of interviewees with relation to views of (a) the meaning of deductive reasoning, (b) the relationships between logical rules inside and outside mathematics, (c) the aspects of deductive reasoning that can be developed through learning mathematics, and (d) the likelihood of mathematics learning to develop deductive reasoning.

INTRODUCTION

The development of deductive reasoning, not only in mathematics, but in general, is stated as a goal of mathematics teaching in many curricula from all over the world (e.g., National Council of Teachers of Mathematics, 2000; Qualifications and Curriculum Authority, 2006). This study aims to examine the approaches of people who are involved in various aspects of mathematics education and logic to the role played by learning mathematics in the development of general[1] deductive reasoning. Following is a brief review of the literature concerning deductive reasoning – in general, in mathematics, and outside mathematics, and the role of learning mathematics in the development of deductive reasoning.

Deductive reasoning

There are various sorts of thinking and reasoning. Among them are association, creation, induction, plausible inference, and deduction (Johnson-Laird & Byrne, 1991). Deductive reasoning is unique in that it is the process of inferring conclusions from known information (called premises) based on formal logic rules, where conclusions are necessarily derived from the given information and there is no need to validate them by experiments[2]. Valid deductive arguments preserve truth in the sense that if the premises are true, then the conclusion must also be true. An example for a common form of deductive inference is the syllogism, which consists of two premises and an inferred conclusion. For instance, All A are B; Some C are A; Therefore, some C are B. No matter what terms we substitute for A, B, and C, the result is a valid deduction. Thus, the following argument is valid: All kinds of music are enjoyable; punk is a kind of music; therefore, punk is enjoyable. Obviously, not all will agree with this conclusion, but the form of the argument assures us that in the case that the premises are true, the conclusion is true as well.

Deductive reasoning and mathematics

Deductive reasoning is most significant in mathematics. And indeed, deductive reasoning is often used as a synonym for mathematical thinking, especially by the formalist school. The formal mathematical-deductive method is defined as starting with undefined terms, and some unproven statements – axioms or postulates. Other mathematical statements (i.e., theorems) are deduced from them using the rules of formal logic, forming a chain of deductions. In the pure formalist approach statements are neither true nor false because they are about undefined terms. Being free from the need to attend to the truth of mathematical statements enables mathematical explorations not available otherwise. Still, mathematics does not remain in the pure formal level. The undefined terms and axioms are often interpreted in connection to the world in which we live, and truth is associated with these interpretations. In this regard, the axioms of a specific mathematical theory are often said to be true and the theorems deduced from them are then also said to be true (Davis & Hersh, 1981). Deductive reasoning is central to mathematics for proving the truth of mathematical ideas, and for recording these ideas. However, it is commonly accepted in recent years that conjecturing, exploration, and creation of new mathematical objects and ideas are seldom done by deductive reasoning. Rather they are based on inductive and intuitive methods (Eves, 1972; Lakatos, 1976; Polya, 1954), similar to the way science is developed.

Deductive reasoning outside mathematics

Since the early days of Greek philosophical and scientific work, deductive reasoning has been considered as a high (and even the highest) form of human reasoning (Glantz, 1989; Luria, 1976). Still, deductive reasoning plays a different role in science than in mathematics. In contrast with modern mathematics, science strives to describe the real world. The scientific process is based to a large extent on inductive reasoning – developing hypotheses based on empirical observations to describe “truths” or “facts” about our world (Freudenthal, 1977; Popper, 1968). Whereas this process has similar characteristics to the way mathematical conjectures are often developed, the stage of providing evidence for the truth of the conjecture is different. Scientific hypotheses, unlike mathematical conjectures, can only be supported – not proven deductively. Nonetheless, deduction is an important tool in science for refuting hypotheses and also plays a major role in predicting and explaining scientific phenomena (Freudenthal, 1977).

Thus, plausible reasoning, and not deductive reasoning, characterizes science as well as other domains, like law and economics (Polya, 1954). Many suggest that everyday activities are even more remote from deductive reasoning (Duval, 2002, Krummheuer, 1995; Toulmin, 1969). In daily life people do not support their claims by a deductive sequence of derivations. Convincing others in the truth of one's claims (or in the rational of one's choices) is the main concern, and not their validity. Thus, substantial arguments (Toulmin, 1969), which do not have the logical rigidity of formal deductions, but are rather more of the plausible type, are often more used, gradually support a statement or a decision, motivated by the need or desire to convince (Perelman & Olbrechts-Tyteca, 1969).

Developing deductive reasoning via learning mathematics

The essential role that deductive reasoning plays in mathematics, on one hand, and the questionable use of deductive reasoning in other fields, on the other hand, raises several issues related to mathematics education. One of them (to which this study relates) is the question of developing deductive reasoning via mathematics learning. Indeed, curriculum guidelines, textbooks and teacher guides in many countries state that mathematics teaching helps students develop their ability to reason logically, and that one of its goals is the development of deductive reasoning, not only in mathematics, but in general. For example, the *Qualifications and Curriculum Authority* (2006) states: "Mathematics equips pupils with a uniquely powerful set of tools to understand and change the world. These tools include *logical reasoning*, problem-solving skills, and the ability to think in abstract ways" (emphasis added). Similar claims are suggested by several researchers (e.g., Clements & Battista, 1992; Morris & Sloutsky, 1998). For example, Polya (1954, p. v) wrote: "Everyone knows that mathematics offers an excellent opportunity to learn demonstrative reasoning". However, Polya himself challenges the role of demonstrative reasoning in real life situations: "Anything new that we learn about the world involves plausible reasoning, which is the only kind of reasoning for which we care in everyday affairs". Later he continues: "The general or amateur student [one who does not intend to make mathematics his life's work] should also get a taste of demonstrative reasoning: he may have little opportunity to use it directly, but he should acquire a standard with which he can compare alleged evidence of all sort aimed at him in modern life" (p. vi). A question is, then, raised – to what extent should the development of deductive reasoning be part of mathematics education? This study asked for the opinions of people who are involved in mathematics education and logic about the connections between mathematics learning and the development of general deductive reasoning.

METHODOLOGY

The research population includes 21 participants. Most of them, 17, belong to at least one of the following groups: junior-high school mathematics teachers, mathematics teacher educators, mathematics curriculum developers, researchers in mathematics education, and research mathematicians. Two other participants are researchers in science education who study logical thinking, and the remaining two are logicians. Individual semi-structured interviews were conducted with each one. The interviews lasted one to two hours, and focused on different issues related to the role of learning mathematics in the development of deductive reasoning. The interviews were transcribed. Using the Grounded Theory method (Glaser & Strauss, 1967) we coded the data from the interviews and generated initial categories, which were constantly compared with new data from the interviews. Based on refinement of the initial categories, we identified core categories, and used them as a source for theoretical constructs. Some of the main aspects that were developed through this process are discussed in this paper: The meaning of deductive reasoning, in general, in mathematics and outside it; the aspects of deductive reasoning which can be developed through learning mathematics; and the likelihood of mathematics learning to develop deductive reasoning.

DEVELOPING GENERAL DEDUCTIVE REASONING VIA LEARNING MATHEMATICS

All 21 interviewees who participated in this study argued that learning mathematics could develop general deductive reasoning. They also pointed out that developing deductive reasoning should be one of the objectives of mathematics education. One interviewee, for instance, when asked whether he thinks that improving deductive reasoning is a goal of mathematics education, replied:

> Eventually the instruction of mathematics has two main objectives. One of them is to train those people who will use mathematics, and the other is intended for those who won't use mathematics afterward – to present an example of deductive reasoning… I think that developing deductive reasoning is a very important aim… It is the role of mathematics teaching (interviewee no. 5, a curriculum developer and a teacher educator).

But what do the interviewees mean when saying that learning mathematics could improve deductive reasoning? How likely it is that learning mathematics will contribute to the development of such reasoning? What do they mean when claiming that improving students' deductive reasoning is one of the goals of mathematics instruction? And what is their approach to deductive reasoning, in mathematics and outside it? Analysis of the interviews reveals that the interviewees provide different answers to these questions and attribute different meanings to the following aspects: the meaning of deductive reasoning, its nature in mathematics and outside it, the aspects of it which can be developed through learning mathematics, and the likelihood of mathematics learning to develop deductive reasoning. Three distinct groups of interviewees were identified, with the members of each group consistent in their approaches to each aspect. Below is a short review of the groups' views, accompanied with a few (because of limits of space) illustrative excerpts from the interviewees.

Group A

Four interviewees belong to group A. They describe deductive reasoning as a process in which one develops a solution to a given problem in a systematic, step-by-step manner. Each step of this process is derived from the previous one, and leads to the next. However, no indication was given as to how a step is derived from its predecessor. These interviewees consider the logical rules inside mathematics as identical to those of outside-mathematics thinking. They view logical rules, both inside and outside mathematics, as systematic principles of thinking, according to which thinking progresses step by step.

These interviewees claim that learning mathematics contributes to the improvement of deductive reasoning in the development of systematic habits of mind. They ascribe this development to the systematic structure of mathematics and to the methodical, step-by-step way of solving mathematical problems. According to them, the development of deductive reasoning occurs spontaneously as a consequence of doing mathematics. Doing mathematics provides experiences in, and thus improves students' deductive reasoning. For example, an interviewee was asked whether learning mathematics could improve deductive reasoning. She replied:

> I think that mathematics improves deductive reasoning, and I think that it is one of mathematics' main goals… I know that generally, as I told you, it will teach him [the student] to think logically and will give him tools to think and a desire to think and to be organized and systematic… Just from learning mathematics, his logical thinking develops in other fields in life as well. But I don't want while teaching, in every new theorem or in every new formula I teach him, to ask myself what kind of systematic tool it provides him with… I don't take each thing and filter it through a 'thinking strainer'… It happens by itself (interviewee no. 11, a senior high school teacher).

Group B

Thirteen interviewees belong to group B. They relate deductive reasoning to an action of inference or validation using rules of formal logic. Whereas group A focuses on deductive reasoning as a systematic, step-by-step process, group B members center on characteristics of the transition from one step in the deductive process to the next: they focus on the logic essence of an inference, on its validness according to logical rules. In addition, while group A focuses on deductive reasoning as means of solving a given problem, group B members refer to it as means of building and validating arguments. They assert that logical rules used in mathematics (i.e. formal rules of inference) are also used outside mathematics, for example, when trying to understand the insurance rights one would have according to different levels of price. However, these interviewees claim that different factors affect deductive reasoning outside mathematics. Thus people apply other, usually 'softer' rules of inference, in addition to the rigorous ones. Two distinct opinions regarding the factors that affect reasoning outside mathematics are found among the interviewees: Six of them (group B1) talk about external conditions, such as uncertainty and complexity of phenomena in nature and society. The other seven (group B2) explain the distractive influence by internal conditions, such as emotions and beliefs.

Group B members claim that learning mathematics could develop habits of argumentation (not necessarily deductive). Mathematics, they claim, because of its particular nature of validation, enables the exposure to deductive justifications and validations. Moreover, its relatively abstract, detached from reality nature can provide students with opportunities to learn and to apply logical validation, without the distractive influence of prejudices and beliefs that exists in life. Thus, for example, emphasizing the meaning of proof or the different functions of statements (e.g., given information, claims to be supported), can contribute to the improvement of students' skills of argumentation, such that are also relevant to outside mathematical contexts. The examples given by those interviewees include providing grounded justifications (even if not deductive) for beliefs and knowledge in daily life, or critically examining of the rationality of claims.

Unlike group A, group B members argue that there is a need for a deliberate intervention in the mathematics instruction in order that mathematics contributes to the development of argumentative skills. Some claim that in order to teach mathematics in a way that will improve these skills, logic should be introduced as a separate unit of study within mathematics. Others suggest various ways of emphasizing deduction constantly and continuously, claiming that in order to develop argumentative skills, there is a need to explicitly teach and practice principles of deduction as an integral part of mathematics lessons, in various situations and problems,

> I think that students who are involved with deductions in mathematics, and whose teacher points at deductive connections and at logical mistakes, can improve their deducing ability, and to like, for example, to look for deductive connections or to identify logical fallacies. I know I like doing it. Even if it is not a real deductive argument, but more of the plausible kind, whoever meets deductions in mathematics will be able to make much more rational inferences in his life, not only intuitive ones (interviewee no. 9, a curriculum developer and a teacher educator).

Group C

Four interviewees belong to group C. Like those of group B, they also see deductive reasoning as an action of inference or validation using logical rules. However, they argue that outside mathematical contexts, we do not or even cannot use the formal logic rules existing in mathematics. One reason for that claim was that the essence of thinking inside mathematics is entirely different from that outside it. Another explanation was that in daily life, as opposed to mathematics, one barely encounters suitable circumstances for using logical rules. Some also argued that even if one encounters such an opportunity, it is not likely that s/he applies them, because in everyday discourse specific argumentative norms exist. According to these norms, the logic of an argument that one builds is neither a necessary condition for understanding nor for accepting the argument.

The interviewees of this group believe that learning mathematics may influence students' general deductive reasoning. However, they find it hard to point out in what ways exactly. Moreover, according to them, even if the possibility of promoting students' deductive skills through learning mathematics does exist, it seems difficult to reach, because of the current demands of the educational system, especially the matriculation exams. For example, an interviewee was asked whether learning mathematics can improve deductive reasoning. He replied:

> It is not that I think it impossible to teach deductive reasoning through mathematics. I believe that mathematics has some influence on this thinking. I just don't know what kind of influence, and can't tell how it could be done. And even if we assume that it is possible to do so, I don't believe it is possible in the present system… How can one teach and learn logical thinking if one is facing the pressure of the matriculation exam? (interviewee no. 4, a researcher in mathematics education and a mathematician).

As these interviewees do not offer an alternative system by which mathematics instruction can promote deductive reasoning, they actually leave the question of promoting it via learning mathematics open and with deep reservation. Table 1 summarizes these findings.

Table 1: Summary of findings

	Meaning of deductive reasoning	Logical rules inside and outside mathematics	What learning mathematics develops in deductive reasoning	The likelihood that learning mathematics improves deductive reasoning	# interviewees n=21
A	Systematic process	Unification	Habits of mind of systematicness	Spontaneity	4
B	Formal logic based inference	Inclusion (external, internal)	Habits of mind of argumentation	Intervention	13 (6,7)
C		Separation	Cannot point out	Reservation	4

CONCLUSIONS

The findings of this study suggest that all its participants view the development of general deductive reasoning as a goal of mathematics instruction. They all assume that to some degree this goal is attainable. However, differences were found among the participants regarding the likelihood and degree of difficulty of achieving this goal. The differences seem to relate mainly to the participants' approaches to deductive reasoning, in general, in mathematics, and outside it[3]: Some of them describe deductive reasoning as a systematic step-by-step approach for solving problems. Being systematic in thinking is one feature of deductive reasoning, which characterizes other kinds of reasoning as well. It is also something that people come across in diverse non-mathematical situations. Likewise, these interviewees consider the logical rules inside mathematics to be identical to those in outside-mathematics thinking. Consequently, these interviewees may naturally point at the simplicity by which the development of systematic habits of mind occurs through learning mathematics. On the other hand, the interviewees who describe deductive reasoning as an action of inference based on rules of formal logic, attribute, as the literature does, complexity to the nature of deductive inference in different domains of life. Accordingly, they consider the development of deductive reasoning through mathematics learning as a complex process that requires deliberate intervention. Moreover, their referring to the aspects of deductive reasoning which can be developed does not relate exclusively to deduction (they refer to argumentation, but not necessarily to deductive argumentation). Some of these interviewees are even not sure whether the process of developing deductive reasoning through learning mathematics is at all possible.

The fact that most interviewees claim that, to some extent, mathematics learning can, and even should, contribute to the development of deductive reasoning, suggests that this issue deserves further attention. In particular, it would be worthwhile to examine in what sense, and under what conditions, learning mathematics develops (as most interviewees claimed) skills of argumentation. Another issue raised by this study is whether specific sub-communities in the community of mathematics educators tend to approach deductive reasoning and its development through learning mathematics in particular ways. Group A includes two teachers and two curriculum developers, one of which is also a teacher educator. There are several other teachers, teacher educators and curriculum developers in the other groups as well. However, all the mathematicians, the researchers in mathematics education and in science education, and the logicians belong to the other groups (B&C)[4]. Indeed, the relative small size of the research population does not allow generalization. Still, it seems worthwhile to study more thoroughly whether there is a connection between the nature of people's professional activities and their approaches towards deductive reasoning and its development through learning mathematics.

NOTES

1. By 'general deductive reasoning' we mean deductive reasoning that is not restricted to mathematics, but can be implemented in other fields as well.

2. This is the classic approach to deductive reasoning, which is also adopted in this paper. There are also other approaches; the main one is based not on formal rules of inference but on manipulations of mental models representing situations (Johnson-Laird, 1999).

3. For an elaboration of these approaches see Ayalon & Even, 2006.

4. A more refined report on the characterization of the approaches of each type of population is in preparation.

REFERENCES

Ayalon, M. & Even, R.: 2006, 'Deductive reasoning: Different conceptions and approaches', In J. Novotnå, H. Moraovå, M. Kråtkå, and N. Stehilkovå (Eds.), *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2 (pp. 89-96). Prague:PME.

Balacheff, N.: 1999, *Is argumentation an obstacle? Invitation to a debate...*, International newsletter on the teaching and learning of mathematical proof. (May/June 1999). Retrieved from: http://www.lettredelapreuve.it/Newsletter/990506Theme/990506ThemeUK.html

Clements, D. H., and Battista, M. T.: 1992, 'Geometry and spatial reasoning', in D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420-464). New York: Macmillan.

Davis, P. J. and Hersh, R.: 1981, *The mathematical experience*. Boston: Birkhäuser.

Duval, R.: 2002, 'Proof understanding in mathematics', Proceedings of 2002 *International Conference on Mathematics: Understanding proving and proving to understand* (pp. 23-44). National Taiwan Normal University. Retrieved from:

Eves, H.: 1972*, A survey of geometry*. Boston: Allyn and Bacon.

Freudenthal, G.: 1977, *Philosophy of science (Units 1-4).* Tel-Aviv: Open University (in Hebrew).

Glantz, I.: 1989, *Thinking in three-directional functioning*. Tel Aviv: Cherikover (in Hebrew).

Glaser, B. G. and Strauss, A. L.: 1967, *The discovery of grounded theory: Strategies for qualitative research.* New York: Aldine.

Johnson-Laird, P. N.: 1999, Deductive reasoning. *Annual Review of Psychology*, 50, 109-135. Retrieved from: http://arjournals.annualreviews.org/doi/abs/10.1146/annurev.psych.50.1.109

Johnson-Laird, P. N. and Byrne, R. M. J.: 1991, *Deduction*. Hillsdale, NJ: Erlbaum.

Krummheuer, G.: 1995, 'The ethnography of argumentation', in P. Cobb, & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229-269). Hillsdale, NJ: Erlbaum.

Lakatos, I.: 1976, *Proofs and refutations: The logic of mathematical discovery.* Cambridge: Cambridge University Press.

Luria, A. R.: 1976, *Cognitive development: Its cultural and social foundations*. Cambridge, MA: Harvard University Press.

http://www.lettredelapreuve.it/Newsletter/990506Theme/990506ThemeUK.htmlMorris, A. K., and Sloutsky, V. M.: 1998. Understanding of logical necessity: Developmental antecedent and cognitive consequences. *Child Development*, 69, 721-741.

National Council of Teachers of Mathematics: 2000, *Principles and standards for school mathematics*. Reston, VA: Author.

Perelman, C. and Olbrechts-Tyteca, L.: 1969, *The new rhetoric: A treatise on argumentation*. Notre Dame, IN: University of Notre Dame Press.

Polya, G.: 1954, *Induction and analogy in mathematics.* Princeton, NJ: Princeton University Press.

Popper, K. R.: 1968, *The logic of scientific discovery*. New York: Harper & Row.

Qualifications and Curriculum Authority: 2006, *National curriculum online.* Retrieved from http://www.nc.uk.net/index.html

Toulmin, S. E.: 1969, *The uses of arguments.* Cambridge, UK: Cambridge University Press.

HOW TO DECIDE? STUDENTS' WAYS OF DETERMINING THE VALIDITY OF MATHEMATICAL STATEMENTS

Orly Buchbinder & Orit Zaslavsky

Dept. of Education in Technology & Science

Technion – Israel Institute of Technology

This article presents an overview of an ongoing study on mathematical reasoning patterns of high school students. The initial findings came out of a study that examined learning processes and student understandings related to the concept of counterexample. More specifically, it examined the ways in which students understand and use counterexamples in mathematics, in the course of studying a special unit designed to foster opportunities for determining the validity of numerous mathematical statements. During the study a number of strategies, which students employed in the process of evaluating the validity of mathematical statements, were identified. These strategies involved a range of underlying cognitive processes that became the main focus of the current research.

BACKGROUND

Student learning and understanding of mathematical proof has been one of the main issues of mathematics education research (e.g. Fischbein, 1982; Hoyles, 1997; Harel, 2002; Mariotti, 2006). Fischbein (1982) addressed the tensions between students' formal and empirical approaches to proof. Hoyles (1997) found that students have difficulties with proof, which derive from their reliance on empirical findings. There is also evidence that students often tend to employ example-based reasoning. By this we refer to justifications that use examples to convince one's self or others regarding a certain assertion (Rissland, 1991; Zaslavsky & Shir, 2005). This is often similar to what Harel (2002) terms empirical proof scheme. In spite of the logical limitations of such reasoning in terms of generalization, it is a useful approach mathematicians often use to develop a 'guts feeling' regarding the validity of mathematical conjectures (Alcock, 2004).

From a logical perspective the use of counterexamples is very simple: one counterexample is sufficient to refute a false universal claim, i.e. claim of the form $\forall x, P(x)$. As such, counterexamples are considered an important tool in the development of mathematics (Balacheff, 1991; Lakatos, 1976). Polya (1973) emphasises the role of counterexamples as an integral part of problem solving strategies, while Michener, (1978) regards counterexamples as one of the basic elements of expert knowledge of mathematics.

Despite the seemingly simplicity of counterexamples, empirical studies indicate that students often posses wrong conceptions associated with counterexamples, their generation and use (Balacheff, 1989; Reid, 2002; Zaslavsky & Ron, 1998). Balacheff (1991) identified several different ways in which students treat counterexamples. For example, many students are reluctant to accept a single counterexample as sufficient proof of a fallacy. Students tend to reject or treat counterexamples as exceptions. Kaur & Sharon (1994) found that many college students limit the domain of examples they check when evaluating an algebraic statement to integers, ignoring negative numbers, fractions and zero.

Some logical aspects of the use of counterexamples appear to cause major difficulties to students (e.g., Helsabek, 1975; Dubinsky et al., 1988). Along this line, Zaslavsky & Ron (1997) observed several difficulties students encounter in generating and using counterexamples, e.g., the inability to distinguish for a given statement between an example that constitutes a counterexample and one that doesn't; or the generation of 'non existing' counterexamples.

THE STUDY

The purpose of the study was to examine and characterize underlying processes in which students engage when dealing with counterexamples, including difficulties they encounter. In particular, it aimed at identifying students' ways of evaluating the validity of mathematical statements (both valid and faulty), with a focus on the role of counterexamples in these processes.

For the purpose of the study, a teaching unit that addresses students' difficulties with counterexamples was especially designed (in two parallel versions adjusted for two different grade levels) and implemented in two classes: top level 10^{th} grade and low level 12^{th} grade. The activities drew on students' prior mathematical knowledge of algebra, geometry and calculus tackling various aspects of counterexamples. The teaching experiment lasted about two months, during which 6 various activities were interwoven throughout the regular curriculum.

The study was conducted in the form of action research (Ball, 2000), in which the researcher served both as developer of the learning environment and as the teacher implementing it. Most of the data was collected, during classroom activities, in the form of audio recordings of students' interactions as they worked in small groups and of whole-group classroom discussions. In addition, written pre and post questionnaires were used, as well as the researcher's journal with field notes and reflections. The questionnaires were used for the purpose of triangulation as an additional source of information about students' conceptions regarding counterexamples.

FINDINGS

The findings suggest that engaging in different kinds of activities that emphasize various aspects of counterexamples, helped students improve their understanding of the notion of counterexample and its use. The same effect was observed in both research groups, regardless of the mathematical level or age of the students. The analysis of students' responses revealed that students came to recognize counterexamples as legitimate tools for refuting false statements; they became more aware of the domain of validity of mathematical universal statements and of the caution needed to avoid overgeneralization of conjectures. In addition, students in both classes improved their content knowledge, as well as their reasoning and communication skills.

One of the most interesting finding was the range of strategies by which students approached the need to evaluate the validity of mathematical statements. These strategies can be described as various paths connecting sequences of points in which decisions need to be taken. Interestingly, similar strategies were observed in both high and low level students. Some recurring paths which we term patterns were identified.

As described above, the students' strategies consisted of sequences of decision making steps. Figure 1 (in the form of a flowchart) presents the various paths students took, including the most common type of reasoning employed at different stages.

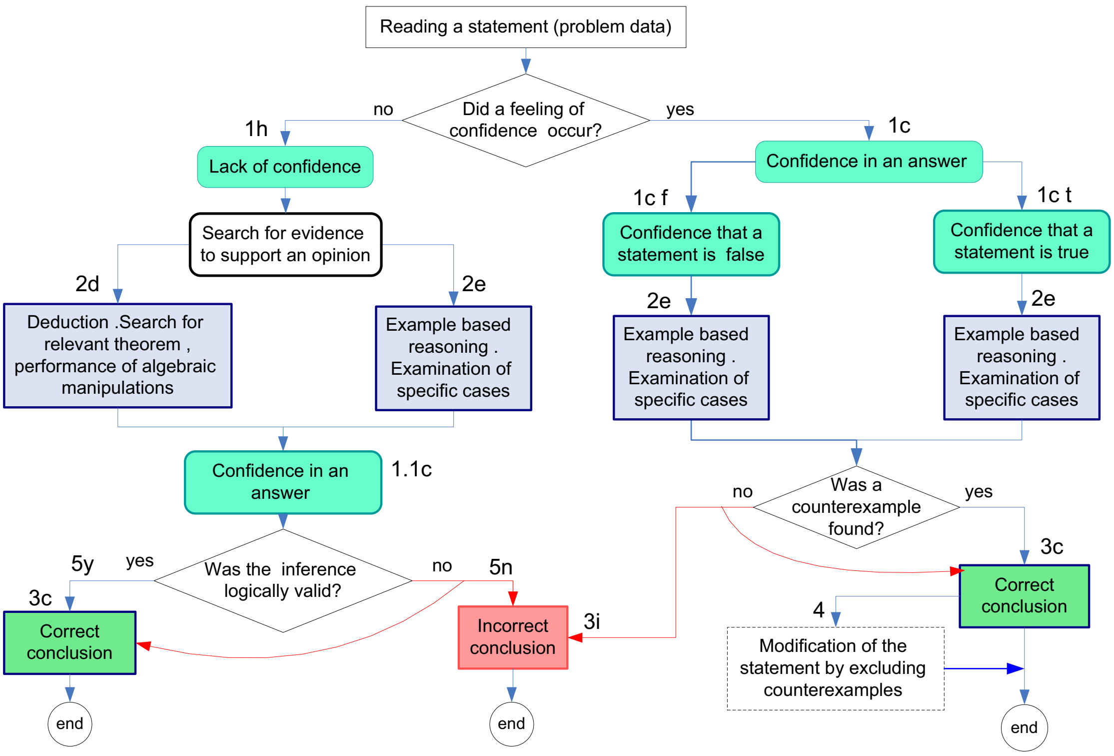


Figure 1: Students' strategies in determining the truth value of mathematical statements.

As shown in Figure 1, the first step students took for a given statement was based on their intuition and sense of confidence. If they felt confident of its truth value (see 1ct & 1cf in Figure 1), they stated their assertion and supported it by example-based reasoning, that is, by examination of specific examples (see 2e in Figure 1). The continuation of the path depended to a certain extent on the truth value of the given statement (True or False) and on the student's initial assertion (Correct or Incorrect). Thus, a student who was confident that a true statement was false, could not find any counterexample to support his assertion; or a student who asserted that a false statement was true, could have found by chance a counterexample that contradicted his assertion. In both cases, the example-based evidence had an effect on the student's final decision (see 3c & 3i in Figure 1), occasionally resulting in students shift to a different decision, sometimes accompanied by a modification of the original statement, to exclude the examples that 'didn't fit' (see 4 in Figure 1).

Students who had no 'guts feeling' in the initial stage (see 1h in Figure 1) expressed hesitation and a need to gather evidence in order to form an opinion. Some turned to example-based reasoning, while others took a deductive approach, by attempting to recall relevant theorems that can help them decide (see 2e & 2d in Figure 1). In either approach, after some time, the students reached a decision and expressed confidence about it (see 1.1c in Figure 1). Clearly, the correctness of their decision depended on the validity of their inferences (see 5y & 5n in Figure 1).

We turn to six examples that illustrate students' reasoning patterns underlying the processes of determining the truth value of a given mathematical statement, along the paths described above. We distinguish between 4 main situations: The **S**tatement may be **T**rue or **F**alse (TS / FS), and the students' initial **D**etermination could be **C**orrect or **I**ncorrect (CD / ID). Examples 1-4 illustrate 4 different cases (TS-CD, TS-ID, FS-CD, FS-ID). Examples 5 & 6 illustrate cases in which students did not come up with an initial assertion about the validity of the statement. In each of the following examples we begin with the statement the truth value of which students were asked to determine.

<u>Example 1</u>(TS-CD):

Statement 1: *The sum of any three odd numbers is an odd number*.

Approach: This is a valid (**T**rue) statement. Many students determined correctly from the start that it is true, without resorting explicitly to a detailed justification. In order to justify their judgment students turned to an investigation of specific examples, which we regard as example-based reasoning (Rissland, 1991; Zaslavsky & Shir, 2005). A typical response was: "Since $(-3)+1+5=3$ and $7+(-13)+(-1)=7$ the statement is always true".

This is a case where students generated a number of examples satisfying the conditions of the statement, and (not surprisingly) did not 'bump' into a counterexample. From a logical point of view, this does not constitute a proof, because theoretically there could be a counterexample that has not been found yet. However, since this statement is true, no counterexample exists.

<u>Example 2</u> (TS-ID):

Statement 2: *The domains of function $f(x)$ and its derivative $f'(x)$ are not necessarily the same.*

Approach: This is a true statement, which some students determined first as false. Students' initial intrinsic feeling was that the statement is false, i.e. they wrongly asserted that the domains of any function *f(x)* and its derivative *f '(x)* must always be the same. In order to support their answer, students checked several examples, and came up with an answer, such as: "This statement is false…we tried some examples... Like $y = \sqrt{x}$ …"

In terms of students' approach, this is similar to Example 1. This answer is particularly interesting, since in the case of $f(x) = \sqrt{x}$, the domains of the function and its derivative are in fact different. This function could have served either as a counterexample to the student's initial decision, or as a proof that the statement is true. It seems like an initial intuitive feeling influenced not only students' choice of inference, but also their perception of the evidence they collected.

Example 3 (FS-CD):

In this case, students' task was to determine whether the given statement is always true, can be true in some cases or never true:

Statement 3: *In order to multiply a number by 10, you just need to write an additional "0" to its right.*

Approach: This is a false statement, which some students wrongly identified as 'always true'. In order to justify their initial assertion, they conducted a short investigation with different numbers, and came up with a counterexample (e.g., 0.4). Students accepted it as a refutation of the statement and modified their assertion. They also made an attempt to adjust its domain of validity by excluding the contradicting examples and refining the statement. Their final answer was something like: "The statement is false, since 0.4 doesn't satisfy it. But, it's true for all numbers larger than 1". It's easy to see that also the new statement proposed by the students is false.

The process of modification of a statement by excluding counterexamples and refining its domain of validity is a rational way of treating a mathematical statement (Lakatosh, 1976). But in this case students missed a crucial step. The validity of the 'new' modified statement needs to be examined. This step was ignored by students in both groups, even by students in high level class.

Ending an evaluation process without checking the validity of a new statement is logically incorrect. On several occasions students arrived at erroneous statements because they failed to check their new conjectures.

Example 4 (FS-ID):

Statement 4: *If two triangles have 2 sides and 3 angles that are equal, then the triangles are congruent.*

Approach: This is a false statement. Students (in the high level class) noticed immediately that the word "respectively" is missing and suspected that the statement is false. In order to refute this statement they initiated an explicit search for a counterexample. When they couldn't find one, students came up with the following sketch claiming it constitutes a counterexample (Figure 2).

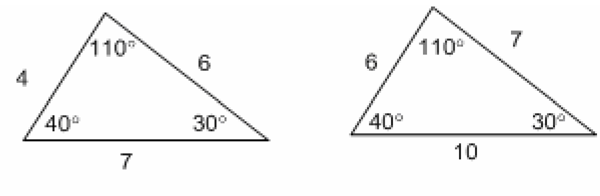


Figure 2: A non-existing 'counterexample' suggested by students

In this case, the students' intuition about the statement was correct, but they were not able to systematically construct a valid counterexample. Instead, they supported their claim by creating what they thought were two triangles that have 5 equal elements (as required) but are not congruent (although in their drawing they look as if they are congruent). However, by imposing too many conditions the resulting 'triangles' in Figure 2 do not exist, thus, cannot serve as a counterexample. At some point, during their discussion, students realized this problem, but were not able to figure out where exactly they went wrong.

<u>Example 5</u>:

Statement 5: *The function* $y = x^4 + 12x + 12$ *never gets negative values*.

Approach: Students had no initial feeling whether the statement is true or false. Thus, they expressed the need to investigate the matter further in order to gather evidence for determining the truth value of the given statement. Students first turned to consideration of special examples (e.g., negative numbers, fractions, 2, -2), that is a bottom-up approach:

Student 1: Listen to me. For this to be negative x must be a fraction.

Student 2: Why?

Student 1: If it is not a fraction, then x^4 is…more. If it's 2, then it's much bigger because…

Student 3: No. Wait. The function never gets negative values. Why? [because] if we take 2, how much is 2^4? [….] and if we take (-2)? 16. So…ok…also not good. What if we take a number smaller than 1?

Another approach was a top-down one. Students tried to retrieve an appropriate rule or theorem that would help them determine whether the given statement is false or true. An example of such an approach can be found in the following reaction:

Student 1: Can't we prove it using derivatives and all the stuff we usually do? It says here "function", and we are trying numbers. It's a function…y is above zero. So. It never gets negative values…

Student 2: Never below zero for any x.

Student 1: No. [only] if we *prove* that it's always above zero, then it's true…

As a result of both approaches, students developed confidence about the statement, i.e., they became convinced that the statement is false. Both kinds of reasoning led students to the right conclusion through logically valid inferences.

Example 6:

In this case, students' task was to determine whether the given statement is always true, can be true in some cases or never true:

Statement 6: *For any real values of* a,b,c *such that:* $a^{-1} = b^{-1} + c^{-1}$*, it follows that:* $a = b + c$.

Approach: This statement is false. Moreover, for any values of *a, b, c*, the two conditions ($a^{-1} = b^{-1} + c^{-1}$ & $a = b + c$) cannot co-exist. Similar to the case described in Example 5, here too students had no initial feeling regarding the validity of the given statement. More precisely, students were uncertain whether or not there are any values at all for *a, b, c* for which the two conditions $a^{-1} = b^{-1} + c^{-1}$ and $a = b + c$ exist. They expressed the need to investigate the matter in order to gather evidence to form an opinion. Students who chose an example-based inductive approach gave responses that were similar to: "$1^{-1} + 1^{-1} \neq 1^{-1}$ ... It's never true, because no such numbers exist. No numbers satisfy this equation."

Students who chose a deductive approach, began by performing some algebraic manipulations on the equation: $a^{-1} = b^{-1} + c^{-1}$, in order to find out whether it has any solutions. A typical response of this kind was: "$\frac{1}{a} = \frac{1}{b} + \frac{1}{c} \Rightarrow a = \frac{bc}{b+c}$ … so the statement is never true."

None of the students completed the task, although the top level students had the algebraic skills needed to do it. It seems that both example-based and deductive approaches were used by students only to gather cues regarding the truth value of the given statement. They searched for evidence that would help them form an opinion and build confidence in it. Once they were confident in their assertion, they ended the work, without noticing that their way of justification and reasoning was incomplete.

DISCUSSION

Our findings point to patterns of students' mathematical reasoning in the context of examining the truth values of mathematical statements that they have not studied beforehand. Some concur with the vast literature on proof (e.g. Harel, 2002; Balachef, 1991; Fischbein, 1987; Zaslavsky & Shir, 2005); particularly, students' reliance on intuitive evaluation of mathematical statements and their rapid use of example-based reasoning. However, there are some unique contributions of this study to our understanding of students' ways of reasoning.

Most studies, concerning students' proof practices focus on the way students prove conjectures, not on the ways they disprove them. Knuth (2002) pointed out that most classroom activities related to proof emphasise its role in validation. Students are often expected to prove results that seem obvious to them. Consequently, it is difficult for students to develop an appreciation of the need to prove. This concurs with Mariotti's view (2006) that if proof does not contribute to knowledge construction through activities that integrate a social dimension, it is likely to remain meaningless and purposeless in students' eyes.

The setting of our study relied to a large extent on the element of uncertainty as a trigger for examining the truth value of mathematical statement, in the spirit advocated by Mariotti. We provided a rich environment for fostering a genuine need for reasoning and revealing students' spontaneous approaches to justification and proof. Their search for convincing evidence was driven by their uncertainty regarding the validity of a statement, rather than by an external requirement to prove. Students had to determine the validity of mathematical statements, produce arguments to support their assertions and communicate their mathematical ideas to their peers. In addition, these arguments became a subject for whole class discussions, eliciting comparisons with arguments that are acceptable, i.e., that are already stated and shared in the mathematics community (Mariotti, 2006).

This special learning environment provided us with opportunities to identify students' natural tendencies and preferences. Thus, we identified strengths and weaknesses of students' inferences. For example, there were little rejections of counterexamples by students, contrary to the findings of Balacheff (1991). On the other hand, students tended to accept statements that they had modified without testing their validity.

We would like to offer another lens through which to examine our findings. In recent years, a number of researches in the psychology of thinking and reasoning have advocated 'dual process' theories of cognition (Evans, 2003; Kahneman, 2002; Stanovich & West, 2000). However, current theories of reasoning propose that the term 'dual process' does not suggest the existence of two distinct systems, but rather two cognitive processes that might reflect different modes of one complex system (Osman, 2003). We would like to apply a dual framework to our findings since it provides useful characteristics of students' cognitive processes, but without making strong assumptions about underlying mechanisms. The word "system" is used here as a broad term for mode or process.

The dual framework contrasts implicit cognitive processes (fast, unconscious, automatic) with explicit processes (slow, conscious, and controlled). The labels "System 1" and "System 2" are associated with these two modes of cognitive functioning (Kahneman, 2002). The framework suggests four ways in which judgment may be made. (1) No intuitive response comes to mind, and the judgment is produced by System 2. (2) An intuitive judgment or intention is evoked and (2a) is endorsed by System 2; or (2b) serves as an anchor for adjustments that respond to other features of the situation; or (2c) is identified as incompatible with a subjectively valid rule, and blocked from overt expression (Kahneman, 2002).

The observed paths of students' reasoning concur with these ways, described by Kahneman. Examples 5 & 6 refer to the option (1), when no initial intuitive feeling regarding the truth value of a statement occurred. Determination whether the given statement is true or false was made by System 2, in other words, through explicit analytical process.

Examples 1-4 refer to option (2), when System 1 was evoked and students got an immediate feeling of confidence regarding the truth value of the statement. This feeling became a subject of further explicit analytical investigation, as part of the function of System 2. In the case of Example 3, students discovered a counterexample that served a basis for correcting their initial response. The initial assertion was overridden by System 2 while a counterexample served as an anchor for modification of the intuitive answer. This is consistent with option (2b) in the described above ways of judgment.

Examples 1, 2 & 4 refer to option (2a), meaning that System 1 came up with an initial response that was endorsed by System 2. In Example 1, this endorsement is justified and students arrived at a correct decision. In Examples 2 & 4 System 2 failed in its function of monitoring the output of System 1. Students' intuitive impression was so powerful, that they did not recognise a counterexample when they saw it (Example 2) or created a non existent counterexample, when they had a strong conviction that a statement is false (Example 4).

In all patterns described above, we witnessed the strong affect of implicit intuitive reactions that guided students' mathematical behaviour. This phenomenon has wide empirical and theoretical support (Fischbein, 1987).

Explicit analytical thinking was also present in students' reasoning which we documented. This can be seen in their: attempted [direct] search for inductive evidence; discovery or construction of counterexamples and when seeming appropriate - modification of statements. These manifestations constitute strong evidence that an analytical cognitive process is present in students' reasoning and is part of their thinking strategies.

Behaviour such as ad hoc modification of a statement and its acceptance without further testing, preliminary termination of investigation, and overgeneralization of inductive evidence, suggests that the strength of an intuitive impression can interfere with analytical cognitive processes. Intuitive cognitive processes may be directing the final judgment, sometimes ignoring the relevant cues or relevant content knowledge.

More research is needed to fully characterise students' strategies in determining a truth value of mathematical statements. Elaboration of those findings in extensive theoretical framework, like the dual process theory outlined here, can contribute to broader interpretation of research findings and better understanding of students' mathematical reasoning.

REFERENCES

Alcock, L.: 2004, 'Uses of Example Objects in Proving', in Hoines, M.J., Fuglestad, A.B.(Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 2*, 17-24.

Balacheff, N.: 1991, 'Beyond a Psychological Approach: the psychology of Mathematics Education'. *For the Learning of Mathematics*, *10*(3), 2-8.

Ball, D. L.: 2000, 'Working on the inside: Using one's own practice as a site for studying teaching and learning', in A. E. Kelly, and R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education.* Mahwah, NJ: Lawrence Erlbaum Associates, Publishers, pp. 356-402.

Dubinsky, E. Elterman, F., Gong, C.: 1988, 'The Student's Construction of Quantification'. *For the Learning of Mathematics, 8*(2), 44-51.

Evans, J. St. B. T., Over, D. E.: 1996, *Rationality and Reasoning*. Hove: Psychology Press.

Fischbein, E.: 1987, *Intuition in Science and mathematics*. Dordrecht, The Netherlands: Reidel.

Harel, G.: 2002, 'The development of mathematical induction as a proof scheme: A model for DNR-based instruction', in S.R. Campbell and R. Zazkis (Eds.), *Learning and Teaching Number Theory: Research in Cognition and Instruction*. Ablex, Westport, CT, pp. 185-212.

Helsabeck, F.: 1975, 'Syllogistic Reasoning: Generation of Counterexamples'. *Journal of Educational Psychology, 67*(1), 102-108.

Kahneman, D. (Nobel prise Lecture, December 8): 2002, 'Maps of bounded rationality: A perspective on intuitive judgment and choice', in *Les Prix Nobel*, Edited by T. Frangsyr, pp. 416-499.

Kaur, B., Sharon, B.H.P.: 1994, 'Algebraic misconceptions of first year college students', *Focus on Learning Problems in Mathematics*, *16*(4), 43-58.

Knuth, E.: 2002, 'Teachers' Conceptions of Proof in the Context of Secondary School Mathematics'. *Journal of Mathematics Teacher Education, 5*(1), 61-88.

Lakatos, I.: 1976, *Proofs and Refutations*. Cambridge University Press.

Mariotti, M. A.: 2006, 'Proof and Proving in Mathematics Education', in Gutierrez, A., Boero, P. (Eds.), *Handbook of Research on the Psychology of Mathematics Education*. Sense Publishers, Rotterdam, Netherlands. 173-204.

Michener, E. R.: 1978, 'Understanding Understanding Mathematics'. *Cognitive Science, 2*, 361-383.

Osman, M.: 2004, 'Evaluation of dual-process Theories of Reasoning'. *Psychonomic Bulletin and Review, 11*, 988-1010.

Polya, G.: 1973, *How to solve it*. Princeton, NJ: Princeton University Press.

Reid, D. A.: 2002, 'Conjectures and Refutations in Grade 5 Mathematics', *Journal for Research in Mathematics Education, 33*(1), 5-29.

Rissland, E.: 1991, 'Example-based reasoning', in J.F. Voss, D.N. Perkins, J.W. Segal (Eds.), *Informal Reasoning and Education*, Hillsdale, NJ: Lawrence Erlbaum, pp.187-208.

Stanovich, K. E., West, R. F.: 2000, 'Individual differences in reasoning: Implications for the rationality Debate', *Behavioral and Brain Sciences, 23*, 645-726.

Zaslavsky, O., Ron, G.: 1998, 'Student's Understanding of the Role of Counter-Examples'. In A. Oliver and K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education, 4*, pp. 225-232.

Zaslavsky, O., Shir, K.: 2005, 'Students' Conceptions of a Mathematical Definition', *Journal for Research in Mathematics Education*, *36*(4), 317-346.

CABRI'S ROLE IN THE TASK OF PROVING WITHIN THE ACTIVITY OF BUILDING PART OF AN AXIOMATIC SYSTEM

Leonor Camargo, Carmen Samper, Patricia Perry

Aprendizaje y Enseñanza de la Geometría (Æ · G)
Universidad Pedagógica Nacional, Bogotá-COLOMBIA

We want to show how we use the software Cabri, in a Geometry class for preservice mathematics teachers, in the process of building part of an axiomatic system of Euclidean Geometry. We will illustrate the type of tasks that engage students to discover the relationship between the steps of a geometric construction and the steps of a formal justification of the related geometric fact to understand the logical development of a proof; understand dependency relationships between properties; generate ideas that can be useful for a proof; produce conjectures that correspond to theorems of the system; and participate in the deductive organization of a set of statements obtained as solution to open-ended problems.

INTRODUCTION

Our research group Æ · G, constituted in 2003, has centred its activity on issues related to the learning and teaching of proof and proving in Geometry. One of its goals is to identify conditions and actions that foster learning to prove in a university level Geometry course. Particularly, we are interested in using Cabri as a mediating tool in the learning to prove process. Our framework is based on ideas exposed in several research agendas (Radford, 1994; Jones, 2000; Laborde, 2000; Mariotti, 2000; Marrades and Gutiérrez, 2000; Healy, 2000), which promote that when geometric construction tasks are linked with the practice of justifying and organising axiomatic systems, the possibility of learning to prove is increased.

Most research on teaching mathematical proof with Cabri focuses on secondary or high school students in a Geometry course. The principal studies focus on the analysis of the roles of proof in the Mathematics curricula, students' difficulties in proving, and teaching experiments to encourage learning to prove (Mariotti, 2006). Many studies investigate how to introduce the students to a theoretical perspective of Geometry, linking geometrical constructing tasks with production of statements to justify, for example, why certain geometric properties of a construction remain invariant when we drag their free objects (Jones, 2000). Other studies advance towards the teaching of proof, analyzing the role of the drag function in helping students look for properties, special cases, counterexamples, etc., that could be related to form a proof (Marrades and Gutierrez, 2000). But studies based on these same aspects with university students, corresponding to the rigorous treatment required at such a level, are insufficient (Marrades and Gutiérrez, 2000).

In this paper, we show how the software Cabri is used when the activity in the class, with preservice mathematics teachers, consists in building a part of an axiomatic system of Euclidean Geometry following, at least partially, the development of a specific system proposed by a mathematician. The examples hereby presented have been chosen because they help us to show how the students participate in the construction of the axiomatic system, transferring knowledge and information obtained using Cabri, for the justification of geometric facts, to the usual context of a written proof. We will illustrate the type of tasks that impel students to recognize the relationship between the steps of the construction and the steps of a formal argument and thus help them understand the logical development of a proof; understand dependency relationships between properties; generate ideas that can be useful for a proof; produce conjectures that correspond to theorems of the system; and participate in the deductive organization of a set of statements obtained as solution to open-ended problems. With our proposal, we hope to contribute new elements about the use of Cabri in the learning to prove, when the task is building an axiomatic system.

RESEARCH FRAMEWORK

We adopt the sociocultural perspective that views learning as "*becoming a participant in certain distinct activities* rather than as becoming a possessor of generalized, context – independent conceptual schemes" (Sfard, 2002, p. 23). What is learnt, in this case, is a distinctive task of the mathematics community, proving, which includes not only actions related to the act of justifying but also actions associated with formulating conjectures, all of which must be theoretically warranted by an axiomatic system.

About the teaching of proof, unlike a direct axiomatic presentation of a system, we are proposing what de Villiers (2004) denominates "rebuilding approach". The content isn't displayed to the students as a finished structure; it is constructed by the apprentice, with teacher scaffolding, trying to create a typical organization. This approach is promoted by researchers like Polya and Freudenthal (cited in de Villiers, 2004) when they declare that students must follow a similar way by which the mathematical content was discovered or invented. A "rebuilding approach" allows a meaningful approximation to the content and creates the conditions that enable students to actively participate in the construction and development of the axiomatic system.

With the purpose of using the "rebuilding approach" of a part of an axiomatic Geometry system, Cabri assumes a central role, as environment that offers the mediation for the construction of meaning of statements that could be theorems of the theoretical system. Specifically, the main ideas that underlie the design of the empirical study and the analysis of data are:

- The possibility of establishing a correspondence between the figure construction tools in Cabri and the properties and geometric relations of the figures constructed in a classic Euclidean Geometry. This allows the introduction of a method of validation, derived from the analysis of the construction process, and to link the steps of a construction that illustrates a theorem with the steps of a deductive argumentation to prove it (Mariotti, 1997).

- The formulation of open-ended problems that give rise to the production of conjectures of the form *if... then...* as a way to take advantage of the diverse exploration tools that the software has. Radford (1994) proposes modifying the theorems that are going to be incorporated in the axiomatic system into statements of the type: conditions that a certain figure must fulfil so that it has such property, thus creating open-ended problems.
- The possibilities of exploring figures in Cabri with the intention of finding properties that help the students elaborate proofs. The use of Cabri helps students look for properties, create auxiliary lines, recognise parts of special triangles or quadrilaterals that could be linked to form a proof (Marrades and Gutierrez, 2000; Laborde, 2000).
- The "soft" or "robust" constructions (Healy, 2000) that lead to the production of diverse conjectures associated to a family of figures which can be organised to form part of the axiomatic system constructed.

THE TEACHING EXPERIMENT

The sample

The teaching experiment has been carried out, during successive semesters, with future secondary school Mathematics teachers in the course *Plane Geometry*, which corresponds to a second semester course of the curriculum. During 10 semesters, the curriculum blends the study of Mathematics with courses in Mathematics Pedagogy and Didactics. The Mathematics courses cover topics of the principal branches of this discipline: Algebra, Geometry, Calculus and Statistics (more or less with the same requirements as expected when the degree is in Mathematics). The study of didactics and pedagogy is centred on the process of teaching and learning Mathematics and on analysing the Math studied in schools. *Plane Geometry* is the second Geometry course that the students take at the University. The first course, *Elements of Geometry*, has been designed to introduce students to the field of Geometry, using an intuitive and informal approach, where the main tools used are ruler and compass. The aim of this course is to help students gain a conceptual frame for future courses; students analyse several important, but isolated, geometrical properties and work on developing skills such as visualisation, conjecturing, communicating and arguing. It is in *Plane Geometry* where they first face the task of formal proving, within an axiomatic system.

The experiment

The teaching experiment has taken place during the 16 weeks of the *Plane Geometry* course, for several semesters. The topics officially included in the course are the usual ones: relations between points, straight lines, planes, angles, properties of triangles, quadrilaterals, circles, and congruency and similarity relations. Some pre-established conditions for the course are: the general task for both teacher and students is the construction of a portion of an axiomatic system of Plane Geometry, through their participation in problem solving tasks and their social interaction; opportunities for the students to engage in the activity of proving are given since the beginning of the course; the teacher's role is to introduce the students into the activity of proving; Cabri is used as a mediation tool that contributes to form a suitable environment for proving.

We have the conviction that when students explore problems with Cabri, they feel confident about the truth of their conjecture, and find important ideas to help them construct a proof. Also, we are looking for a meaningful approach to the concepts and relations studied, in an environment in which students have the opportunity to work together: (i) exploring geometrical properties; (ii) finding regularities while they solve problems; (iii) making conjectures; (iv) formulating justifications about geometrical facts and, (v) organising those ideas and justifications in a particular axiomatic system. Instead of having the teacher expose the axiomatic system directly, we want the students to make connections between empirical and theoretical forms of working and, to participate as a community, whose task is learning to prove while building an axiomatic system for Plane Geometry, reason why not always the entire course topics are covered during the semester.

The course has always been taught by one of the authors of this paper. One of the researchers was present in all the class sessions during the first semester of 2004, taking field notes and making an audio register of the general discussions and the group work, which were later transcribed. During the successive semesters, the events in the classroom continued being object of analysis and revaluation. The team kept on meeting periodically to decide what events of those classes should be registered and analysed, and to design the tasks to be used. Designing open-ended problems that are related to the axiomatic system so far constructed and are useful for the activities of conjecturing and proving, has been a complex task, in spite of the many beautiful problems that exist in dynamic geometry but whose proofs require geometric knowledge which is far beyond that which is included in our system. The study of all the fragments gave rise to the identification of the examples that are hereby reported to illustrate how we use the software to help build an axiomatic system.

CABRI'S ROLE AND SOME STUDENT RESULTS

Using Cabri to understand the logical development of a proof

One of the main norms established in the class, with respect to the type of proof accepted, is the logically organized argument using definitions, axioms and theorems previously known and accepted by all. To help the students understand the logical development of a proof we use the idea, raised by Mariotti (1997), concerning the relationship between the theoretical process of a proof and the organization required for the construction of a figure that illustrates the theorem.

Statement	**Justification**	**Construction in Cabri and related steps in the proof**
1. *A*, *B*, and *C* are non collinear points.	Given	Draw three non-collinear points *A*, *B*, and *C*. (1)
2. $\overleftrightarrow{AB}$ exists.	Line Postulate	
3. $\overline{AB}$ exists.	Definition of segment	Draw $\overline{AB}$. (3)
4. Let *M* be the midpoint of $\overline{AB}$.	Midpoint Theorem	Find midpoint *M* of $\overline{AB}$. (4)
5. $\overleftrightarrow{CM}$ exists.	Line Postulate	Draw $\overleftrightarrow{CM}$. (5)
6. $CM = \text{r}$, $\text{r} > 0$.	Distance Postulate	
7. Let 0 and *r* be coordinates of *C* and *M,* respectively.	Ruler Placement Postulate	Using compass, circle or measure transfer (requires finding the length of $\overline{CM}$ directly or indirectly (6)), find *D* on $\overrightarrow{CM}$.(8)
8. Let *D* be point on $\overrightarrow{CM}$ such that coordinate of *D* is 2r.	Ruler Postulate	Verify that *M* is midpoint of $\overline{CD}$. (10, 12)
9. $0 < \text{r} < 2\text{r}$.	Property of real numbers	
10. *C-M-D.*	First Betweeness Theorem	
11. $CM = \mid \text{r} - 0 \mid = \text{r}$, $DM = \mid 2\text{r} - \text{r} \mid = \text{r}$.	Ruler Postulate	
12. *CM* = *DM.*	Transitive Property	
13. *M* is midpoint of $\overline{CD}$.	Midpoint definition	
14. $\overline{AB}$ and $\overline{CD}$ bisect each other.	Definition of bisector	

Figure 1

For example, after the first postulates, definitions and theorems of the axiomatic system are established, we proposed the following problem: *Given three non collinear points A, B and C, show that there exists a point D such that* $\overline{AB}$ *and* $\overline{CD}$ *bisect each other*. All students did a similar construction in Cabri, as is described by a group of students, which we have transcribed as the third column in the table (Figure 1). The teacher then asked them to compare the steps of the construction with the statements and justifications of a proof, which led them to include the proof step number (given in parenthesis) after each sentence and which helped them understand the connections between the proof and the construction.

Using Cabri to help students develop ideas for a proof

The following example, which was designed following suggestions given by Radford, and Marrades and Gutierrez, illustrates how interaction with Cabri, in the process of studying an open-ended problem, provides information that is useful for a proof. This experience took place when the students had finished studying the topics related to triangles and quadrilaterals. The problem we asked them to solve was: *"In isosceles triangle ABC, determine the position of the point P, on the base of the triangle, so that the sum of the distances from P to the congruent sides of the triangle is a minimum. Justify your answer."*

The students started the exploration after constructing the isosceles triangle, locating a point *P* on the base, constructing the perpendicular segments from *P* to the congruent sides and calculating their lengths. They dragged point *P* and very soon realised that the sum is invariant. They wrote conjectures such as: *"It doesn't matter where the point is; the sum of the distances is constant"* (Figure 2).

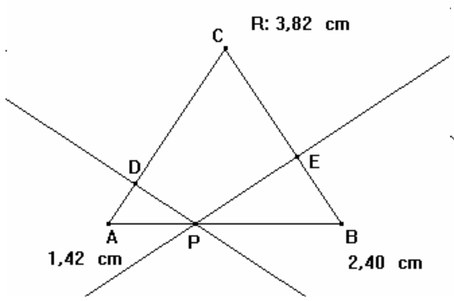


Figure 2

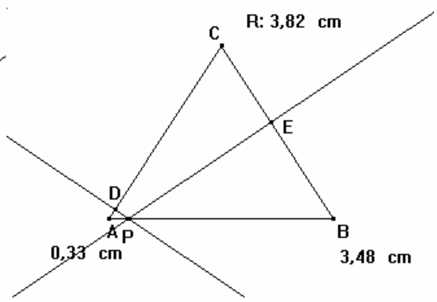


Figure 3

Some students moved *P* until it coincided with point *A* and noticed that $\overline{PE}$ became the altitude of the triangle relative to $\overline{BC}$ (Figure 3). The exploration of a "limit case", locating *P* on one of the endpoints of the segments, shows that they were looking for ideas, based on critical situations, to support their conjecture. The above discovery was very important because it gave them a geometric reference for the sum. It wasn't only a constant value but a very special value: the height relative to the congruent sides. The students then began to draw auxiliary lines, searching for a way to obtain triangles or quadrilaterals, which could be used to prove why the sum is equal to the height, because the elements used in prior deductive proofs had been corresponding parts of congruent triangles or properties of special quadrilaterals.

After some exploration, a group of students discovered how to make good use of the parallelism between the altitude to $\overline{AC}$ and $\overline{PD}$ (Figure 4a). They constructed $\overline{PQ}$

perpendicular to $\overline{BG}$ which led them to the congruency of $\overline{DP}$ and $\overline{GQ}$. This was definitely the key step to be able to prove that $\overline{QB}$ was congruent to $\overline{PF}$, which follows from the congruency of ΔPQB and ΔBFP (Figure 4b).

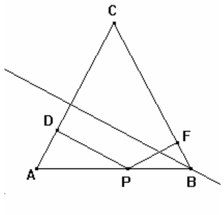


Figure 4a

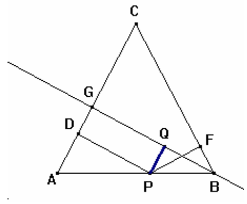
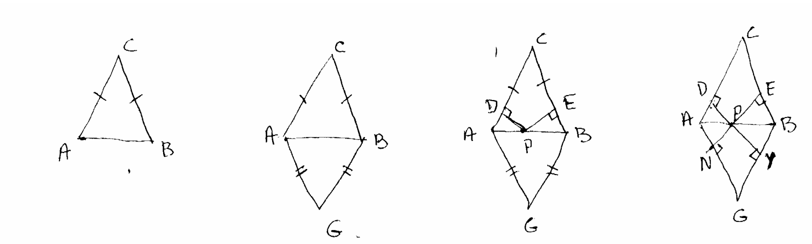


Figure 4b

Another group of students used the symmetry tool of Cabri, which corresponds to a concept not included in the constructed axiomatic system, to reflect the triangle with respect to its base. A careful exploration of the resulting figure, dragging points, taking measurements, helped them realize that the reflected image was congruent to the original triangle. Eventually, this led to the construction of a proof, which one of the students presented to the class, using the following sequence of figures which he drew on the blackboard. The idea underlying their proof is that quadrilateral ACBG is a parallelogram and therefore the distance between opposite sides is always the same, and that ΔDAP ≅ ΔNAP, so DP + PE = NP + PE.



(a) (b) (c) (d)

Figure 5

Using Cabri to create situations where students obtain enough results to collectively organize them as a part of an axiomatic system

For this task, the teacher posed a problem, chosen because of the amount of conjectures students can produce which are related to the situation involved. Using geometric open-ended problems, whose solution permits diverse conjectures about a specific theme, with the support given by Cabri to explore, conjecture and verify results, has shown itself to be a way to involve students in the activity de Villiers (1986) has denominated as descriptive axiomatization.

After formulating a set of geometric properties and relations, as conjectures students feel sure about, with the teacher's guidance, the community organizes the results into a part of the axiomatic system. The teacher's role is essential because she has to design the best path to examine each conjecture, avoid circular reasoning, obtain economical definitions, and establish the correct connections between the results, looking for mathematical coherence. We will report two instances of such situations.

With respect to triangle properties, sometimes students don't understand the need to prove "evident" propositions as, for example, the Isosceles Triangle Theorem: *The base angles of an isosceles triangle are congruent*. We use Cabri to explore interesting properties that require the "evident" theorems, as a means to incorporate them into our axiomatic system. For example, instead of asking the students to prove the above theorem, we ask them to solve the following problems:

> Draw ΔMOP. K is a point on $\overleftrightarrow{MP}$. (a) When is $m\angle OKP > m\angle OMK$? (b) In ΔOKP, when is $OK > OP$?
>
> What is the relationship between the type of triangle and the property: two congruent altitudes?

When students explore these problems with Cabri they find that: (i) the external angle of a triangle is larger than the internal nonadjacent angles; (ii) when two sides of a triangle aren't congruent, then the longest side is opposite to the largest angle; (iii) when two sides of a triangle aren't congruent, then the largest angle is opposite to the longest side; (iv) two of the altitudes of an isosceles triangle are congruent.

Statements (i) and (iv) are properties that students feel they can prove, but when they try to prove the latter, they realise they need the Isosceles Triangle Theorem. Thus we create the necessity of formally including it in the axiomatic system. They also need two new triangle congruency criteria: SAA Congruency (side–angle–angle) and HL (hypotenuse– leg). These criteria can be proved using result (i). The path followed to construct this part of the theory is: Isosceles Triangle Theorem → (i) → SAA congruency criteria → HL congruency criteria → (iv). Another sequence followed is: Isosceles Triangle Theorem → (ii) → (iii). The students' experimental results are organized in a deductive way.

With respect to quadrilateral properties, the problem we use, is the following one:

> What is the relation between the type of quadrilateral and the property: a diagonal bisects the other one?

This is an open-ended problem without a single answer. There are a lot of quadrilaterals which have that property, in a strict sense, and they aren't special quadrilaterals. However, students, unconsciously or deliberately, add other properties to the given one, giving rise to a variety of answers. If students explore the situation using a soft construction (Healy, 2000) and centre their attention on having the diagonal satisfy the given condition and another one, they will formulate conjectures of the form: "if the diagonals of a quadrilateral are… then the quadrilateral is…". They can, for example, imagine that both diagonals bisect each other, and therefore create a parallelogram. Conjectures, like the following ones, are established: "*If a diagonal of a quadrilateral bisects the other diagonal, and both of them are* *perpendicular, then the quadrilateral is a kite*"; "*If both diagonals bisect each other and they form right angles, the quadrilateral is a rhombus*".

But this problem also gives rise to another set of different conjectures. If they begin their exploration, using a robust construction (Healy, 2000) and, following their intuition, construct a special type of quadrilateral to examine the relationship between the diagonals, they will state conjectures of the form: "If a quadrilateral is … then the diagonals…". For example, "*if a quadrilateral is a parallelogram, its diagonals bisect each other*", conjecture that becomes Theorem 1 of the chain that includes all the parallelogram properties which they have discovered or arise in their attempt to prove the conjecture. The axiomatic deductive approach usually employed to introduce the content of the class, changes. The teacher decides which conjecture should be examined first to begin the deductive chain, incorporating other conjectures.

Using Cabri to help students understand dependency relationships between properties

In accordance with Laborde (2000) and Jones (2000), when students explore open-ended problems and write conjectures, they can have difficulty in recognising the properties used in their constructions that conform the "real" hypothesis of their conjecture and therefore guarantee the property discovered. They then write conjectures that don't correspond with the construction that they have made. When students are asked to review the construction process, explain their procedure, we help them grasp all the conditions exposed in the problem, realise whether they have imposed additional or restrictive properties, and understand the dependency relationships involved and, therefore, the logic behind a statement of the form if… then…

For example, when James, a student, was solving the problem related to quadrilateral properties, mentioned above, he wrote the following conjecture: *"In a quadrilateral, if a diagonal bisects the other diagonal, then the quadrilateral is a parallelogram".* Only when teacher asked him to review the construction did he understand that he was using a more restrictive property, because he included the condition that both diagonals bisect each other.

FINAL REFLECTIONS

Constructing an axiomatic system means not only studying the different elements that conform it: definitions, axioms and theorems, but also understanding how, legally, the latter elements are incorporated into the system, through valid proofs. Being able to construct a proof requires the comprehension of the dependency relations between geometric properties, the ability to visualise auxiliary constructions that permit connections with known facts, the conviction that proving is the only legitimate way to include geometric facts in the system, and the genuine desire to carry on the task.

Students' participation in the proving process increases when they are encouraged to propose new ideas, make conjectures, and listen to and participate in the mathematical arguments of their partners as members of an inquiry community of practice. The teacher has the responsibility to design interesting tasks to promote the mathematical activity of his or her students, establish several opportunities for proving and stimulate a rich interaction so students can move from a peripheral place in the community to the core of it. The use of Cabri to explore open-ended problems allows students to take active part in discovering geometric facts by themselves, and incorporating them, and those discovered in the process of proving the conjectures, into an axiomatic system.

REFERENCE

de Villiers, M.: 1986, 'The role of axiomatisation in mathematics and mathematics teaching', Research Unit for Mathematics Education (RUMEUS). South Africa: University of Stellenbosch.

Healy, L.: 2000, 'Identifying and explaining geometrical relationship: Interactions with robust and soft Cabri constructions', *Proceedings of PME 24*, Hiroshima, Giapponi, 1, 103 – 117.

Jones, K.: 2000, 'Providing a foundation for deductive reasoning: students' interpretation when using dynamic geometry software and their evolving mathematical explanations', *Educational Studies in Mathematics* 44, 55 - 85.

Laborde, C.: 2000, 'Dynamic geometry environments as a source of rich learning contexts for the complex activity of proving ', *Educational Studies in Mathematics* 44, 151 - 161.

Mariotti, M.A.: 2000, 'Introduction to proof: the mediation of a dynamic software environment ', *Educational Studies in Mathematics* 44, 25 - 53.

Mariotti, M.A.: 2006, 'Proof and Proving in Mathematics Education', in A. Gutiérrez and P. Boero (eds.), Handbook of research on the psychology of mathematics education, Sense Publishers, Rotterdam, The Netherlands, pp. 173-204.

Marrades, R., and Gutiérrez, A.: 2000, 'Proofs produced by secondary school students learning geometry in a dynamic computer environment', *Educational Studies in Mathematics* 44, 87 - 125.

Radford, L.: 1994, 'La enseñanza de la demostración: Aspectos teóricos y prácticos', *Educación Matemática*, 6, 3, 21-51.

Sfard, A.: 2002, 'There is more to discourse than meet the ears: looking at thinking as communicating to learn more about mathematical learning', *Educational Studies in Mathematics*, 46, 13-57.

SOME REMARKS ON THE THEOREM ABOUT THE INFINITY OF PRIME NUMBERS

Ercole Castagnola*, Roberto Tortora**

**Nucleo di Ricerca Didattica, Università Federico II, Naples, Italy*

***Dipartimento di Matematica e Applicazioni, Università Federico II, Naples, Italy*

Abstract. *The famous Euclid's theorem on the infinity of prime numbers represents a typical case of difficulties for students. In this work we present some reflections and proposals to contrast such difficulties, focused on: a) the problem of proofs by contradiction – in this case viewed as inessential – also in relation with the dychotomy potential/actual infinite; b) a comparison between the current proof and the original Euclid's one, especially for its potential influence on the building of algebraic language; c) the opportunity of privileging students' exploratory activities as necessary steps toward the construction of the proof, and the chances that a wise use of technologies offer to this exploration.*

INTRODUCTION AND THEORETICAL FRAME

It is widely recognized that students encounter several difficulties in understanding and producing mathematical proofs. The problem is somewhat specific in Geometry, where the role of pictures can be seen as a guide to reasoning. But a large part of theorems in Geometry appear as self-evident, so that a proof hardly seems necessary, while in a few cases proofs are difficult, with the result that students often resigns themselves to learn by hearth without any awareness of the arguments. On the contrary, in Arithmetics some meaningful properties can be selected, at the same time simple and non-trivial, therefore very suitable for introducing students to proofs. This is the case of Euclid's famous theorem on the infinity of prime numbers.

In this paper we want to stress some particular features of this theorem, from historical, epistemological and didactical points of view. In particular we want: a) to analyze the influence of a comparison between the original proof and two modern versions of it, on the development of students' linguistic competences (the use of algebraic symbols); b) to show how the difficulties of students in understanding a proof by contradiction, can, in a sense, be neglected, since it is neither strictly necessary nor effective to proceed in this case in an indirect way; c) to stress the role of technologies in assisting the heuristic stage as a necessary step before the proof, both for motivation and for a (partially) autonomous construction of the proof itself.

Many authors have variously emphasized the importance of using history (in particular, original sources) in Mathematics Education (Fauvel & Van Maanen, 2000), (Katz, 2000), (Furinghetti & Radford, 2002), (Castagnola, 2002a). It is not a case that in the last years many texts and materials have been specifically devoted to teachers: for instance, (Berlinghoff & Gouvêa, 2004), with a lot of references, and (Katz & Michalowicz, 2005); or websites like (1), a real mine of historical information; or *Convergence*, an online magazine, (website (4)), where mathematics, history and teaching interact. The introduction of a historical dimension reaches many goals: it humanizes the image of mathematics and helps in modifying the current view of mathematics as made of continuous progresses, showing a lot of sudden turning-points, wrong paths and blind alleys, which gives meaning to otherwise boring students' efforts; and it offers materials to develop intuition, particularly when presented using modern symbols, verbal expressions and cultural tools, instead than, according to Recapitulation Principles, those employed by ancient authors.

A different problem concerns students' difficulties in using algebraic language for abstracting and generalizing (Radford, 2000). Such difficulties are still more evident when a proof is involved (Mariotti, 1998), in particular a proof by contradiction.

Calculators, in particular the graphic-symbolic ones, are widely recognized as precious tools in school practice (Castagnola, 2002b). First of all, they free both teachers and students from the risk of "getting lost" in cumbersome calculations, allowing to turn attention to problems, at the same time more meaningful from a mathematical point of view and closer to the complexity of real world (Kaput, 2002), (Paola, 2006). Moreover, technological tools, bearing an "embodied intelligence", can be seen as powerful means to facilitate objectification and generalization of mathematical concepts (Radford, 2003), and to overcome some students' rigidities ("the prime numbers are only 2, 3, 5" or "the *really existing* numbers are only small integers"). Finally, in Mathematics Education a combined use of history and technology has already taken into account (Castagnola, 2004).

THE THEOREM ON THE INFINITY OF PRIME NUMBERS

The theorem on the infinity of prime numbers is one of the most famous and of the most "beautiful" in the history of mathematics. Several proofs have been produced (see for instance the website (3)), but the best known, modelled on Euclid's original proof, is surely the most easily understood, a striking example of simplicity and elegance. In spite of that, this proof appears much more obscure for students than we could think at first sight. A deep and careful analysis of the proof and of its didactical implications is presented in (Polya, 1973). After that, many authors have focused their attention on the difficulties involved in the contradiction argument employed in the proof, e.g. (Reid & Dobbin, 1998). Other authors have underlined the logical subtleties, all but easy to be understood, involved in such kind of reasoning (Antonini, 2003), (Antonini & Mariotti, 2006); or the necessity to enter an "imaginary" world, where the usual rules of logic can be put in doubt (Leron, 1985). (For more references, see the quoted papers). In particular, Leron notes how the "distance" between the assumption *a contrario* and the conclusion causes the total loss of all the constructions performed in the intermediate steps, erroneously perceived as meaningless.

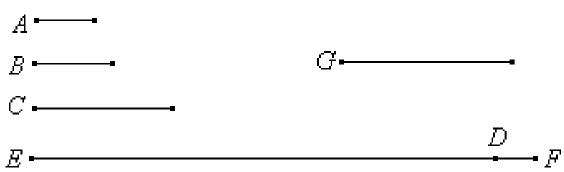
We believe that this kind of difficulties constitute a very serious problem. But since, in the specific case of our theorem, reasoning by contradiction does not seem to us anyhow necessary for the proof, we don't enter here into this subject, which surely deserves further attention, as we intend to do elsewhere. Instead, we prefer to devote our attention to other aspects. In particular, we believe that a big difficulty is connected with the idea of infinity, another one with the use of algebraic notations. Moreover, we think that comparing the actual proof with the original Euclid's version can help students to overcome such difficulties. Another decisive help comes from technological tools, today easily available in classwork.

We want to begin comparing three versions of the proof: the original Euclid's one (Proposition 20, book IX of *Elements*), as reported in (Heath, 1956); the "modern" (1925) version of the same author; and that used today in mathematics texts.

I. Euclid's version.

*"***PROPOSITION 20.** ***Prime numbers are more than any assigned multitude of prime numbers.***

Let *A*, *B*, *C* be the assigned prime numbers; I say that there are more prime numbers than *A*, *B*, *C*. For let the least number measured by *A*, *B*, *C* be taken, and let it be *DE*; let the unit *DF* be added to *DE*. Then *EF* is either prime or not.



First, let it be prime; then the prime numbers *A*, *B*, *C*, *EF* have been found which are more than *A*, *B*, *C*.

Next, let EF not be prime; therefore it is measured by some prime number [VII. 31]. Let it be measured by the prime number G. I say that G is not the same with any of the numbers A, B, C. For, if possible, let it be so. Now A, B, C measure DE; therefore G also will measure DE. But it also measure EF. Therefore G, being a number, will measure the remainder, the unit DF: which is absurd. Therefore G is not the same with any one of the numbers A, B, C. And by hypothesis it is prime.

Therefore the prime numbers A, B, C, G have been found which are more than the assigned multitude of A, B, C. Q.E.D." (Heath, 1956, v. 2, p. 412)

II. Heath's version

"The number of prime numbers is infinite.

Let *a*, *b*, *c*, … *k* be any prime numbers. Take the product $abc\cdots k$ and add unity. Then $(abc\cdots k + 1)$ is either a prime number or not a prime number.

(1) If it *is*, we have added another prime number to those given.

(2) If it is *not*, it must be measured by some prime number [VII. 31], say *p*. Now *p* cannot be identical with any of the prime numbers *a*, *b*, *c*, … *k*. For, if it is, it will divide

> $abc \cdots k$. Therefore, since it divides ($abc \cdots k + 1$) also, it will measure the difference, or unity: which is impossible.

Therefore in any case we have obtained one fresh prime number. And the process can be carried on to any extent”. (Heath, 1956, v. 2, p. 413)

III. A typical today version

There exist infinitely many prime numbers.

Let us suppose that all the prime numbers are the following: $p_1, p_2, \ldots, p_n$. The purpose is to prove that there is a prime number not included in this list. For that, consider the natural number $M = p_1 \cdot p_2 \cdots p_n + 1$ and examine the two alternatives:

Case 1. If M is prime, then it is certainly a “new” prime not included in the previous list, because it is greater than each number $p_1, p_2, \ldots, p_n$.

Case 2. If M is composite, then it has a prime divisor q. We say that q does not belong to the initial list of prime numbers. In fact, if $q = p_k$ for some k, then q would divide both M and $p_1 \cdot p_2 \cdots p_n$ and therefore also their difference $M - p_1 \cdot p_2 \cdots p_n = 1$. But the prime number q cannot be a divisor of 1. This *contradiction* implies that q is different from every p_k and hence it is the new prime we were looking for.

TWO EPISTEMOLOGICAL QUESTIONS

It is very interesting to compare the three proofs. The substance of reasoning is evidently the same, but many meaningful differences leap before our eyes, concerning the meaning of the concepts involved, the “sense” of infinity, the language employed, and so on. (Many interesting comments on Euclid’s “style” are reported in the historical website (2)). As an example, Euclid’s notion of prime number is different from the nowadays accepted one: “*A **prime number** is that is measured by a unit alone*.” (Definition 11, Book VII). Thus, Euclid, like the overwhelming majority of our students, does not consider among the possible divisors of a number the number itself (a divisor must be smaller than the number that it divides). This allows an interesting discussion on the evolution of mathematical definitions (see, for this, (Paola, 2000), (Zaslavsky & Shir, 2005)).

Of course, a detailed analysis of all the differencies between the proofs would bring us very far. Here we want to focus our attention on two particular points: the roles of the reasoning by contradiction and of infinity in the proof of the theorem, and the evolution of algebraic linguistic tools used to denote numerical variables.

The roles of the proof by contradiction and of infinity

By comparing the different proofs, but also their statements, we can note that Euclid doesn’t mention directly the *infinity* of prime numbers. His conception of infinity is *potential*: whatever collection of prime numbers we start with, there is always another prime number not included in it (and the proof shows in what way “to build it”); i.e. prime numbers are always *more* than any established quantity of them.

It is clearly a way of perceiving the infinity more familiar to students; in fact, it corresponds to the very early way of understanding natural numbers as an infinite collection, in accordance with the fact that for every natural number n, big as it can be, there is a bigger one: the *successor* of n. In our opinion this is the only way of conceiving the infinity at the beginning of secondary school. Well, if we state the theorem as Euclid did, then we realize that the proof is direct and even "constructive"; and that nothing prevents us to conclude the proof saying: "Therefore the prime numbers are infinitely many".

The need for the proof by contradiction, whose length – the length of permanence in the "absurd" world – can however be reduced (Leron, 1985), rises on the contrary from the fact that the subtle, and very awkward for students, concept of actual infinity is employed, whose definition is, among other things, given by negation: a set is infinite if it is *not* equinumerous to any initial segment $\{1, 2, …., n\}$ of the set N of natural numbers[1]. To deny such a property we need a double negation: we suppose, by contradiction, that the prime numbers are finitely many, therefore it is not true that doesn't exist an n for which they can be put in a one-to-one correspondence with the set $\{1, 2, …., n\}$; therefore such an n exists and this allows to represent the prime numbers as $p_1, p_2, \ldots, p_n$. Then the proof goes on till the conclusion. It is quite evident that such a way to present the result makes it uselessly involved, by bringing logical and linguistic subtleties in the foreground, and hiding the substance and the constructiveness of the main argument.

On the development of algebraic notations

Let us compare the ways by which the first prime numbers are denoted in the three proofs. In Euclid's proof 3 prime numbers are considered, denoted by the 3 first letters of the alphabet. The proof is than carried out in a way that suggests that if, instead of 3, we had used any number, the result should have been the same. A particular case is treated, but "we see" that it has a general value. The use of the number 3 in Euclid (by the way, it would be amusing to ask why the choice turns just to 3, but this is another talk) is similar to the use and the drawing of a "generic" triangle to argue about any triangle, a question on which many authors have discussed, from Kant onwards, see for instance (Lolli, 2005). This way of proceeding is surely a little "bug", if we see it with the eyes of modern rigour (it is likely that Euclid were aware of this, but he had not at his disposal a more rigorous linguistic tool), but it is also the way we proceed very often also today in mathematical communication, evidently because the greater concreteness of the particular case yields greater effectiveness. Moreover, it stimulates the ability to integrate intuition and reasoning and to control and keep distinct what is specific from what is general.

[1] Not to speak of the deep theoretical problems underlying the definition of an infinite set: as it is well known, the definition is not unique, and the Dedekind's one (a set is infinite if it is equipotent to a proper subset), is equivalent to the first one only if we accept the axiom of choice. The questions involved are by far too challenging for students.

In Heath's version the first prime numbers are denoted by a, b, c, … k, the first letters of the alphabet. The "analogy" between the alphabetical ordering and the order of natural numbers is still kept, but the trick of the dots allows to directly treat the case of any number. Also this linguistic solution can be criticized: after all, k is the 11^{th} letter of the (English) alphabet, but it indicates, on the contrary, any position in the alphabet (evidently 11 seems a big enough number to assume this role). But such a criticism is expression of an excessive pedantry, since in this case (as, and perhaps better, than in Euclid's version) no misunderstanding is possible: as a matter of fact, this type of notation is everyday systematically used without any trouble.

The notation in the third version of the proof is completely different. The first prime numbers are represented by p_1, p_2, …, p_n, where indexes and dots are used to give an account of the indefinite amount of involved elements. There is no doubt about the superiority of this notation, the result of a long evolution and of more and more urgent demands of rigour in the history of mathematics. But we do raise some doubts on the opportunity of using this notation at school, or at least we wonder if it is correct to propose this sophisticated form of language with too much confidence and without the necessary care and graduality. Probably it could help to introduce previously the *list* (an ordered set of elements), an important data structure of computer science, widely used in many scientifical contexts, for instance in statistics.

Coming to students' behaviour, we know that they tend to see in the use of letters only a shortened way to describe some property. For instance they interpret without any difficulty an expression like $A = ½·b·h$ as the formula for the area of a triangle. It is more difficult for them to use an expression containing letters as a tool for abstraction and generalization (Radford, 2000).

In fact, students feel the modern notation in the proof of our theorem too involved, and in general prefer Euclid's proof. This can be observed whenever the proof is proposed by the teacher, and, if they choose by themselves the symbols to represent the situation, almost no one uses a notation similar to the modern one, while Euclid's or Heath's notations appear often. Moreover, we have not to forget that doesn't exist any formula with n as a variable to represent the n^{th} prime number, contrarily to what happens for simpler sequences. For instance it is less difficult (but by no means trivial) to accept the symbol $2k–1$ to represent the general odd number: the reason is that the sequence 1, 3, 5, …, of odd numbers is easily recognised to be generated by natural numbers, by subtracting a unit from the double of each of them, so the symbol $2k–1$ looks exactly as the expression of such a procedure.

We can resume our discussion saying that the proof "with indexes" doesn't convey anything more then the classic one. By this, we are not saying that indexes shouldn't be used (they are useful and sometimes necessary, for example for lists), but only that we must carefully arrive to this point and do not overlap the effort of a proof to that of a too subtle and not strictly necessary use of linguistic tools.

THE ROLE OF TECHNOLOGY IN THE EXPLORATORY PROCESS

According to a constructivistic point of view, we believe that a mathematical result can come in a class only after an exploratory process. In our case, as in other ones, this approach enhances motivation and understanding of the statement of the theorem and of its proof; and, as we will see, it offers also the possibility to touch other topics, to formulate conjectures, to discover properties. But, in order to be able to carry the exploration far enough, technology turns out to be an essential instrument. In this section we illustrate the main lines of a widely experimented didactical path inspired to the above principles. In the classwork the exploratory activity is always intertwined with readings from original sources, and is performed by individual work and collective discussions.

A first activity consists of trying to understand how prime numbers are arranged among natural numbers. For instance, we can build a table collecting the number of primes in each century from 1 to 1000, like the following one (Burton, 2005, p. 383):

Interval	1 - 100	101 - 200	201 - 300	301 - 400	401 - 500	501 - 600	601 - 700	701 - 800	801 - 900	901 - 1000
Number of primes	25	21	16	16	17	14	16	14	15	14

By inspection of this table (and, if necessary, of larger ones, to be found on catalogues or on websites like (3)), we note that prime numbers, though irregularly, tend to become rarer and rarer. It is known (and can be shown to students) that for any number n, it is possible to find a sequence of n consecutive natural numbers which are all composite: for instance, the n numbers $(n+1)!-(n+1)$, $(n+1)!-n$, …, $(n+1)!-3$, $(n+1)!-2$. Moreover, since programs of symbolic calculation like DERIVE and MAPLE contain, in their library of functions, the function $\pi(x)$ that tells how many primes are less than or equal to x, it is possible to graph $\pi(x)$ using bigger and bigger values of x: the graph seems to become more and more "horizontal".

So, the observation of both tables and graphics highlights a phenomenon for which there are two possibilities: either prime numbers somewhere disappear from the sequence of natural numbers, and hence they are finitely many, or for every prime p it is possible to find a greater one, and hence they are infinitely many[2]. The theorem we are considering justify itself as the answer to this dilemma.

Now the problem naturally arises: how a number like $p_1 \cdot p_2 \cdots p_n+1$ came into Euclid's mind? This gives the opportunity of opening a discussion on the question: "If a finite number of primes are given, how can I build another prime not already in

[2] Perhaps we can take here the opportunity of speaking about asymptotes in a non-conventional way. Or, if we are working with young students, it is possible (or even suitable) to riconsider the topic some years later, proposing to them to approximate the function $\pi(x)$ with a function $f(x)$ regular enough, namely with continuous first and second derivatives. Students should conclude that the first derivative has to be non-negative and the second one negative, without ruling out the possibility that $f(x)$ becomes definitively constant. By the way, we know that the function $\pi(x)$ is asymptotic to the function $g(x) = x/\ln(x)$, and that $g'(x) > 0$ for $x > e$ and $g''(x) < 0$ for $x > e^2$.

the list?" So, the construction in Euclid's proof can again be preceded by an exploration. There are many available procedures: for instance, we can carry on the following one. Let's start from the prime number 2 and build the number $a_1 = 2+1 = 3$, which is prime. In the second step build $a_2 = 2\cdot3+1 = 7$, prime. From 2, 3 and 7, obtain $a_3 = 2\cdot3\cdot7+1 = 43$, prime. At the next step we get: $a_4 = 2\cdot3\cdot7\cdot43+1 = 1807 = 13\cdot139$. Both 13 and 139 are "new" primes; we could use both, but taking only the smaller, we obtain: $a_5 = 2\cdot3\cdot7\cdot43\cdot13+1 = 23479 = 53\cdot443$. And so on…

Otherwise we can follow the more "known" path: $b_1 = 2+1 = 3$, which is prime; $b_2 = 2\cdot3+1 = 7$, prime; $b_3 = 2\cdot3\cdot5+1 = 31$; $b_4 = 2\cdot3\cdot5\cdot7+1 = 211$; $b_5 = 2\cdot3\cdot5\cdot7\cdot11+1 = 2311$, all prime numbers; $b_6 = 2\cdot3\cdot5\cdot7\cdot11\cdot13+1 = 30031 = 59\cdot509$. And so on.[3]

This process actually gives more and more new prime numbers. We can use a symbolic calculator (here we are using *TI-89 Titanium*) to overcome the lenghtness and difficulties of calculations but also to distinguish between the two possible cases for b_i, since the command `factor` allows to easily establish if a given number is prime or composite (see Figure 1). When the calculation is not assisted by a powerful tool, it is quite sure that only the first case is noticed, since the first composite value of $p_1\cdot p_2\cdots p_n+1$ is too big. On the contrary, by the aid of a calculator, the exploration can go on without difficulties, to reach for instance the case shown in Figure 2. In our opinion, this is a simple and meaningful example, to see how a calculator can be a really useful tool in helping students to understand and build a meaning.

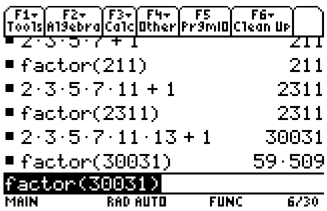


Figure 1

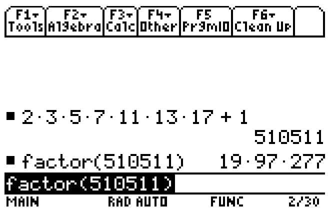


Figure 2

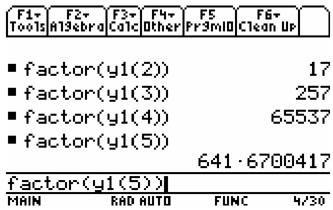


Figure 3

It is important to stress that the observation of a finite number of cases can never replace a proof, but it allows only to do some *conjecture*, to be confirmed or disproved. History tells us how that behaviour can be misleading: it is enough to recall the well known example of Fermat who in 1640 enunciated the conjecture "All the numbers $F_n = 2^{2^n}+1$ are prime" (n any natural number). Let's still use a symbolic calculator to examine the conjecture. We insert in *Editor* (where functions can be defined) the function y1(x) = 2^(2^x)+1. Using the command `factor`, we discover

[3] The numbers b_i are interesting in themselves. As observed, the first five of them are all prime numbers, whereas b_6, b_7, b_8 are not. In (Burton, 2005) many interesting facts are reported: for instance till today (2005) only 19 primes have been identified in the sequence (the largest, discovered in 2000, that is $p_1\cdot p_2\cdot\ldots\cdot42209 + 1$, has 18241 digits), while all the other b_i's for $p \le 120000$ are composite. And nobody knows whether there are finitely or infinitely many primes of the form b_i. Well, knowing about simple problems still unsolved is always a fascinating stimulation for students.

that the conjecture is actually false, showing (see Figure 3) that F_5 is not prime, but is the product of two primes: $F_5 = 2^{32}+1 = 641 \cdot 6700417$[4].

CONCLUSIVE REMARKS AND FURTHER DEVELOPMENTS

In the previous section we have suggested a possible classroom path for the proof of the theorem. One of the two authors has experimented for years in his classes such a path, with different developments and deepenings, according to class contexts and circumstances. We think that the whole experience gives evidence to the goodness of the suggested approach, whereas no specific didactical situation or students' work does it adequately. This is the reason why, – but also due to space limits of this work, – we don't give detailed reports or comments on specific events.

In our opinion two problems would deserve further deepening. The first one concerns proofs by contradiction. Following the opinions of some logicians (Lolli, 2005), we guess that many theorems in school curricula, usually proven by this technique, can also be proven in a direct way, slightly modifying, if necessary, their statements. Then the problem would turn into a linguistic one, namely to show how any implication can be expressed in an equivalent way by its contrapositive. We intend to come back to this problem in a forthcoming work.

The second problem concerns infinity, and its two facets as potential or actual infinity. Obviously, on this topic all has already been said from a conceptual point of view. But we think that the discussion is still open on how and when and why the notion of infinity occurs in school in its two forms. The theorem of prime numbers is an important moment, but it isn't the only one and we think that any possible deepening of this problem would be interesting. We will take care also of this question in the next future.

References

Antonini, S.: 2003, *Dimostrare per assurdo: analisi cognitive in una prospettiva didattica*, Doctoral thesis, Dipartimento di Matematica, Pisa University.

Antonini, S. and Mariotti, M.A.: 2006, 'Reasoning in an absurd world: difficulties with proof by contradiction', in *Proc. of the 30th PME Conference*, Prague, 2, 2-65.

Berlinghoff, W.B. and Gouvêa, Q.: 2004, *Math through the Ages*, *A Gentle History for Teachers & Others*, The Math. Association of America, Washington, D.C.

Burton, D.M.: 2005, *Elementary Number Theory*, McGraw Hill, Singapore.

Castagnola, E.: 2002a, 'Learning to teach algebra: an Italian experience with reference to history', *Proc. of the II Intern. Conf. on the Teaching of Math.*, Heronissos, Greece.

[4] Leonhard Euler (1707-1783) found the result in 1739, of course without technological tools. His method can be read for instance in (Dunham, 1992) or in (Øre, 1988). Here we have another example of an open problem: today it is conjectured that all the numbers $2^{2^n}+1$ are composite for $n \geq 5$, that is there are only 4 Fermat primes. This conjecture has been verified till $n = 30$ (Burton, 2005).

Castagnola, E.: 2002b, 'The functionality of CABRI and DERIVE in a graphic calculator', *Proc. of the Intern. Conf. "The Humanistic Renaissance in Math. Education"*, Palermo, 80-86.

Castagnola, E.: 2004, 'From Euclid to Descartes, to … Cabri. Using history and technology to explore mathematics', *Proc. of HPM 2004*, Uppsala, 167-174.

Dunham, W.: 1990, *Journey through Genius: The Great Theorems of Mathematics*, John Wiley & Sons, USA.

Fauvel, J. and Van Maanen, J. (eds): 2000, *History in mathematics education*. ICMI Study, Kluwer, Dordrecht.

Furinghetti, F. and Radford, L.: 2002, 'Historical Conceptual Developments and the Teaching of Mathematics: from Phylogenesis and Ontogenesis Theory to Classroom Practice', in L. English (ed.), *Handbook of International Research in Mathematics Education*, L. Erlbraum Associates, Mahwah, NJ, 631-654.

Heath, T.L.: 1956, *The Thirteen Books of Euclid's Elements*, Dover, New York [reprint of the II edition, Cambridge University Press, 1925, London].

Kaput, J.: 2002, 'Implications of the Shift from Isolated, Expensive Technology to Connected, Inexpensive, Divers and Ubiquitous Technologies', in F. Hitts (ed.) *Representations and Mathematics Visualization*, Mexico, 80-109.

Katz, V. (ed.): 2000, *Using history to teach mathematics. An International perspective*, The Mathematical Association of America, Washington, D.C.

Katz, V. and Michalowicz, K.D.: 2005, *Historical Modules for the Teaching & Learning of Mathematics*, CD-rom, The Math. Ass. of America, Washington, D.C.

Leron, U.: 1985, 'A direct approach to indirect proofs', *Educ. Studies in Math.* 16, 321-325.

Lolli, G.: 2005, *QED. Fenomenologia della dimostrazione*, Bollati Boringhieri, Torino.

Mariotti, M.A.: 1998, 'Introduzione alla dimostrazione all'inizio della scuola secondaria superiore', *L'insegnamento della Matematica e delle Scienze Integrate* 21B-3, 209-252.

Øre, O.: 1988, *Number Theory and Its History*, Dover, New York.

Paola, D.: 2000, 'Le definizioni: dalla parte degli studenti', *L'insegnamento della Matematica e delle Scienze Integrate* 23A-B, 6, 562-600.

Paola, D.: 2006, 'Nuove tecnologie e studio delle grandezze che variano', *L'insegnamento della Matematica e delle Scienze Integrate* 29A-B, 4, 347-356.

Polya, G.: 1973[2], *How to solve it*, Princeton University Press, Princeton, NJ.

Radford, L.: 2000, 'Signs and meanings in students emergent algebraic thinking: a semiotic analysis', *Educational Studies in Mathematics* 42, 237-268.

Radford, L.: 2003, 'Gestures, Speech, and Sprouting of Signs: A Semiotic-Cultural Approach to Students' Types of Generalization', *Mathematical Thinking and Learning* 5(1), 37-70.

Reid, D. and Dobbin, J.: 1998, 'Why is proof by contradiction difficult?', in *Proceedings of the 22th PME Conference*, Stellenbosch, South Africa, v. 4, 41-48.

Zaslavsky, O. and Shir, K.: 2005, 'Students' Conceptions of a Mathematical Definition', *Journal for Research in Mathematics Education*, 36:4, 317-346.

Websites

(1) http://aleph0.clarku.edu/~djoyce/java/elements/toc.html
(2) http://www-groups.dcs.st-andrews.ac.uk/~history
(3) http://www.utm.edu/research/primes/
(4) http://convergence.mathdl.org

PROOFS PROBLEMS IN ELEMENTARY NUMBER THEORY: ANALYSIS OF TRAINEE TEACHERS' PRODUCTIONS

Annalisa Cusi, Nicolina A. Malara

Dipartimento di Matematica, Università di Modena e Reggio Emilia

Abstract: Our study involves a group of pre-service middle-school [1] *teachers attending educational and training courses at our university. The aim is to classify their behaviour in solving a problem that requires the proof of a statement in elementary number theory not easy to be formalized. This study highlights mental blocks in those who are not able to create an algebraic model for the problem and widespread difficulties related to the impact of abilities both in translating algebraic expressions into the algebraic code and in interpreting algebraic expressions built during the construction of the proof in order to get to the thesis.*

INTRODUCTION

The Italian mathematics curriculum's strong influence on students' failures in constructing proofs is unquestionable. Indeed, the Italian teaching tradition focuses on training students to reproduce proofs and does not devote time to students' autonomous construction of proofs. We are convinced that students should be encouraged to construct important mathematical facts and investigate problems starting from compulsory school, especially in the 6th, 7th and 8th grades. In Italy, more than 80% of lower secondary school mathematics teachers are not mathematics graduates, and their cultural background makes it very difficult for them to carry out this kind of activity.

Recent studies suggest that the teaching of the concept of proof should be promoted in pre-high school grades (Stylianides e Stylianides, 2006b). Nevertheless, in order to promote students' proving abilities, it is necessary for teachers to be able to autonomously manage the solution of proving problems. We agree with Jones (1997) when he writes that "the most-qualified trainee teachers may not necessarily have the specific kind of subject matter knowledge needed for the most effective teaching". Therefore, an investigation on the correlation between teachers' educational background , their conceptions and the different approaches they adopt towards proof problems is necessary. Moreover, we need to highlight the main difficulties they meet, so that they can become aware of both potential and limitations of their own mathematical background. In this way, it will be possible for teachers to develop the content knowledge that can allow them to promote argumentation and proof in their classrooms (Stylianides and Stylianides, 2006a).

Therefore, we decided to work with a group of trainee teachers [2] attending educational and training courses for the teaching of mathematics in middle-school: our aim was to study the impact of abilities in formal coding and in interpreting algebraic expressions on proofs' production, with particular relation to the transition between argumentation and proof. Boero et alii (2002) compared argumentation and proof as linguistic products, stressing that the language adopted is one of the most

relevant points of discrimination between them. In particular, they observe that “the understanding and production of mathematical proofs belong to the literate side of linguistic performances and require prior deep linguistic competence”.
Our hypothesis is that the rupture between argumentation and proof gets deeper due to a lack of activities of both translation from verbal to algebraic language and interpretation.
The analysis we propose here has a double value:
1) It highlights the influence of educational background on the choice of a proving strategy as well as on the problems related to the development of a proof;
2) It represents a moment in a formative “route”, proposed to trainees in order to make them: a) understand the importance of coordinating verbal and algebraic language in the development of proofs; b) compare proof strategies developed through both the verbal and the algebraic register; c) reflect on potentialities and limitations of different strategies (Afterwards we will show what kind of theoretical tools we referred to in order to promote trainees reflections on these aspects).
In this paper we focus on a problem posed to trainees. We chose it because of its particular textual characteristics and because of the kind of proof it implies. In fact, its solution could be “easy” to be found from an intuitive point of view, but its formalization is quite complex. Before presenting an analysis of this problem, we will sketch out a synthesis of some studies about the subject of proof.

THEORETICAL FRAMEWORK

Issues related to the meaning of proof and to its functions have been deeply analysed both from the mathematicians’ community’s and from teachers and mathematics education researchers’ points of view (Hanna 2000; Hersh 1993; Thurston 1994).
Investigating students’ difficulties in producing proofs and searching for reasons underlying their failures or successful results are considered to be crucial issues (Hoyles 1997; Weber 2001). Some researchers have identified proof schemes through which the different attempts carried out by students in developing a proof can be classified (Alcock and Weber 2005; Harel and Sowder 1998). Harel and Sowder, in particular, subdivide proof schemes into three main classes, each of them further divided in subcategories: *external conviction proof schemes* (produced by students who think that “ritual and form constitute mathematical justification”), *empirical proof schemes* (produced by students who “validate, impugn or subvert conjectures by appeals to physical facts or to sensory experiences”), *analytical proof schemes* (produced by those who are able to “validate conjectures by means of logical deductions”).
Other researchers outlined theoretical models for students’ difficulties in proving and suggested possible strategies to promote an appropriate attitude by students in dealing with proofs (Moore 1994; Weber 2003).
In the last decade, studies about students’ production of algebraic proofs in the elementary number theory field intensified also due to the space given to proof in the curriculum of some countries (England is an example). From a didactical point of

view, the construction of algebraic proofs creates problems: in fact, as it is stressed by Barnard and Tall (1997), while algebraic manipulation requires to carry out "sequential procedures in which each mathematical action cues the next", a proof also requires the ability of making choices. Furinghetti and Paola (1997) highlighted the presence of a double "shadow effect" when students face proofs of statements in elementary number theory: a "shadow effect" due to algebra, which prevents students from using their arithmetic knowledge in their attempts of proving, and a "shadow effect" due to arithmetic, which leads students to view only numerical checks as proofs and prevents them from making generalizations explicit.
Analysis of students' proofs of statements in elementary number theory highlighted students' difficulties not only in translating familiar numerical concepts (such as "to be even" or "to be odd") from verbal to algebraic language, but also in deducing all the possible information that an algebraic expression brings with it (Barnard and Tall, 1997).
Some mathematics education scholars propose an approach to algebraic language also related to the development of reasoning in the proving process, referring to natural numbers as a suitable environment for those activities that favour a transition from argumentation to proof through the use of algebraic language (Boero & al. 1995; Malara 2002; Friedlander, Hershowitz, Arcavi, 1989; Sadovsky, 1999).

PRESENTING THE PROPOSED PROBLEM

The problem at stake is the following: *"Suppose that a is a non null natural number. If a is divisible neither by 2 nor by 3, then* a^2*-1 is divisible by 24"*. This problem is taken from the textbook, aimed at 15-16 years old students, "Matematica come scoperta" ("Mathematics as discovery") by G. Prodi (1979). This textbook was thought and written in a research-based environment and it is still very innovative.
We made hypotheses about possible difficulties related to the interpretation of the problem's statement and to the choice of the proof strategy.
As regards difficulties related to <u>text interpretation</u>, we point out that this problem is different from those typical tasks students are exposed to because of its linguistic formulation: the hypothesis is, in fact, expressed by a negation (non-divisibility by 2 and 3). Another element of difficulty in the interpretation of the text is related to the fact that the thesis refers to an element (a^2-1) different from those the hypothesis "talks about" (a).
Other difficulties depend on the <u>choice of the proof strategy</u>. A verbal "approach" [3] to the proof of this statement is based on considerations that concern the concept of (non) divisibility by 2 and by 3 and on considering the relation that involves the dividend, the divisor, the quotient and the remainder of a division. It requires a good control of the implications of the hypothesis, in particular a clear view of the properties of both quotient and remainder of the division by 2 and by 3. The verbal proof develops from the identity $a^2-1=(a-1)(a+1)$: i. ($a-1$) and ($a+1$) are even because a is not-divisible by 2; ii. one among ($a-1$) and ($a+1$) is also divisible by 4 because they are two consecutive even numbers; iii. one among ($a-1$) and ($a+1$) is

divisible by 3 because a is not-divisible by 3; therefore iv. $(a-1)\cdot(a+1)$ is altogether divisible by 24.

Through an *algebraic* approach one can be led to consider also the formalization of the relation between the terms involved in the division of a by 6, besides the formalization of the similar relations that can be inferred from the division of a by 2 and by 3.

The possible algebraic proofs are more complicated than the verbal one because they require the formal translation of the hypothesis, a distinction between different cases, the use of suitable syntactic transformations and, above all, the interpretation of the new expressions produced with relation to the problem.

Therefore we can imagine that an algebraic "approach" to this problem is likely to bring about obstacles for students, in particular related to the translation of divisibility properties in terms of the Euclidean algorithm. Control of the syntactic equivalence of expressions having different senses is fundamental, because it allows one to highlight properties that emerge more clearly in some expressions than in others (Arzarello, Bazzini, Chiappini, 1995).

The choice of this problem is related to the fact that an algebraic "approach" to its solution makes it possible to highlight trainees' flexibility in both formalization and interpretation of formal expressions and also their possible mental blocks.

METHODOLOGY

The problem was given to 54 trainees with different university backgrounds (27 mathematics graduates, 3 physics graduates and the remaining 24 biology, geology, chemistry and natural sciences graduates) in the initial phase of the Mathematics Education training courses. Trainees were supposed to solve, in 45 minutes, the assigned problem, describing the different proving strategies they tried to follow in the solution process and pointing out both obstacles and difficulties they met.

In analysing their protocols, we divided trainees into two groups according to their backgrounds: mathematics or physics graduates and trainees with a degree in other sciences. Our analysis is made through a double lens. In fact, we look at trainees' protocols as products, classifying the proofs they gave making reference to Harel and Sowder (1998)'s classification of proof schemes. At the same time, we analyse their protocols from the point of view of the difficulties which come out in the proving processes. In this second analysis we look at difficulties caused by: problems related to the coordination between the verbal and algebraic registers in carrying out the proof; inability to formalize compound statements; inability to interpret formulas resulting from a syntactic transformation; presence of possible mental blocks.

RESULTS

We divided trainees' productions into six main groups according to the highlighted proving strategies and to Harel and Sowder's proof schemes (we singled out those categories which adhere to the examined proofs). The following table summarizes the proof schemes' frequencies in our trainees' protocols.

Table 1: Frequencies of proof schemes with relation to educational backgrounds

Harel and Sowder's Classification		**Typologies of protocols which adhere to categories**	**Frequencies of proof schemes with relation to educational backgrounds**	
Categories	**Subcategories**		Maths-Physics	Other sciences
External conviction proof schemes	*Symbolic proof schemes*	Algebraic proofs which highlight syntactical and/or interpretative problems in managing formal expressions	**9/30**	**8/24**
	Ritual proof schemes	Proofs in which trainees attempt to apply well-known proving procedures (such as reasoning by induction and proof by contradiction) characterised by serious logical mistakes	**3/30**	/
Empirical proof schemes	*Perceptual proof schemes*	Verbal proofs in which reference to erroneous perceptions about numerical properties is made explicit	**2/30**	**2/24**
	Inductive proof schemes	Proofs based on numerical examples only	**1/30**	**14/24**
Analytical proof schemes	*Intuitive-axiomatic proof scheme*	Verbal proofs in which reference to correct numerical properties is made explicit	**2/30**	/
	Symbolic transformational proof schemes	Correct algebraic proofs	**13/30**	/

The table highlights that *Trainees with "other scientific degrees"* definitely prefer verification through numerical examples (14 out of 24). Those who try to use the algebraic code (only 8 out of 24) get stuck because of their difficulties in manipulating and understanding the algebraic expressions they formulate. The remaining trainees with "other scientific degrees" try, not successfully, to prove the statement using a verbal approach. Those who try to construct a not merely inductive proof also turn to numerical examples. In fact they use numerical examples in their proofs when they want to show the correctness of their assertions or when they are not able to manipulate some expressions. In this context, this attitude hides insecurity. With reference to trainees who verify the statement through numerical examples only, we think it is important to group their protocols according to whether they show consciousness about the limits of their productions or not. In this regard, we can say

that most trainees (10 out of 14) are aware that numerical examples are not enough to prove the statement, so they point out their inability in finding an algebraic representation of the regularities observed through examples. Four trainees, however, think that verifying a property for some numerical cases only is enough to prove it.
As regards "aware" trainees, these are some significant claims made by some of the trainees after having shown that the property holds for some simple numerical examples: *"I can guess that it's true, but I'm not able to formalize it in mathematical terms, so I'm not able to prove it"; "I can't express numbers through symbols"*.
As regards "not-aware" trainees, there are some important claims made by trainees who believe they have correctly proved the statement: *"the property is valid when a=5, 7, 11 so, if we substitute a for 13, we'll find a multiple of 24"*.
Almost all *mathematics and physics graduates* (25 out of 30) prefer an algebraic "approach". Four of them try to give a verbal proof (2 unsuccessfully). Only one of them chooses the inductive scheme. We were negatively surprised by the high number of trainees with a degree in mathematics and physics who failed in the proof of the statement (15 out of 30; 12 of them chose the algebraic strategy) because of their inability to organize a proof strategy or to complete the proof.
We observe the worrying fact that a high number of trainees are not able to master algebraic expressions (41 out of 54, precisely all the trainees with "other scientific degrees" and 17 out of 30 trainees with a degree in mathematics or physics). Some of them openly declare their difficult relationship with algebraic language, others display a rejection of mathematics when they choose not to use any algebraic formalism, others display their difficulties through their unsuccessful attempts to master the algebraic expressions they formulate.

A ZOOM ON PROVING PROCESSES

In the following we will show the <u>main difficulties highlighted by our analysis</u> of trainees' protocols, distinguishing between trainees with a degree in mathematics or physics and trainees with other scientific degrees. We will start from the latest because of their greater weakness in the development of formal reasoning.

Trainees with "other scientific degrees"

Among the main problems highlighted by our analysis of protocols, we would like to stress two kinds of difficulties in particular:
1) <u>Difficulties in correctly interpreting the text of the problem and in inferring properties related to hypothesis and thesis</u>. These are some of the typical mistakes:

1a) A lot of trainees draw wrong conclusions about the set of numbers which satisfy the properties required for *a*. For example, a widespread interpretation makes some trainees conclude that *a* must be a prime number;

1b) Some trainees display that they do not know exactly what to prove. A trainee, for example, writes confidently that it is impossible to prove the statement because *"Every number which is divisible by 24 is also divisible by 2 and by 3"*, confusing the properties of *a* with the properties of a^2-1.

2) Difficulties in interpreting and managing algebraic expressions. Typical mistakes are related to the process of formal coding and to the interpretation of expressions after the syntactic treatment. These are some examples of the most representative mistakes, that we propose following the development of the proof:

2a) Those who try to factorize a^2*-1* stop after having observed that *(a-1)(a+1)* is divisible by 2 and by 3 and they are not able to infer other simple properties, such as the divisibility of *(a-1)(a+1)* by 4.

2b) Almost no trainees are able to formulate the properties of *a* in algebraic language, and in particular the non-divisibility of *a* by 3. Some of them try to solve the problem interweaving algebraic and verbal language. One trainee, for example, expresses the non-divisibility of *a* by 2 and by 3 in this way: *"a/2≠integer and a/3≠integer for every integer greater than or equal to zero"*. Others translate the properties of *a* into expressions with a completely different meaning, as we will see afterwards in a reported example.

2c) Those who try to prove the statement show that they do not know how to deduce the properties of a^2*-1*. One trainee, for example, starts his proof with the formalization of the thesis and writes $a^2 - 1 = 24n$. He immediately stops here because he is not able to further interpret what he has written applying the hypothesis.

In support of the latter remarks related to difficulties in the interpretation of algebraic expressions, we propose an *example of an unsuccessful attempt to give a formal proof, not supported by a real understanding of the algebraic expressions*:

The trainee observes, without any motivation, that every number which is divisible neither by 2 nor by 3 can be written as $(3+2)+n$, where *n* is an even number. Afterwards, he carries out some calculations, writes $(3+2+n)^2 - 1 = 24 + 10n + n^2$ and, in order to prove that this expression is divisible by 24, he sets up the equation $24 + 10n + n^2 = 24h$. Then he manipulates the equation, obtaining the equality $n = 24\frac{h-1}{10+n}$. Finally, he uses this last equality to assert that *"n is divisible by 24"*.

Among the observations that we can make about this proof, we think it is important to stress the total lack of control in understanding and managing both the algebraic expressions that he constructs and the related properties. He is not able to translate the properties of *a* into formal language; he sets the condition that the expression obtained for a^2*-1* satisfies the required property, confusing hypothesis and thesis; he obtains an equality in which *n* is not even made explicit and draws from it erroneous conclusions about the properties of *n*; he draws conclusions about *n* instead of drawing conclusions about a^2-1.

Trainees with a degree in mathematics or physics

Most of those trainees who fail in their proof display: 1) difficulties in the translation form verbal to algebraic language, closely connected with a clear lack of knowledge about elementary numerical properties; 2) difficulties in understanding and managing formal expressions.

With reference to point 1, the main mistakes in translation are related to trainees' inability to formalize concepts such as multiple or divisor, and to use the Euclidean relation between dividend, divisor, quotient and remainder of a division:

1a) Some trainees are not able to make the hypothesis explicit. The following claim clearly shows this problem: *"It's difficult to codify in symbols the concept of not being divisible by 2 and not being divisible by 3"*;

1b) A lot of trainees try to use the Euclidean relation for representing the non-divisibility of a by 3 or by 6, but they do not consider the variability of the remainder. For example, some of those who deduce from the hypothesis the non-divisibility of a by 6 translate this property into the equality $a=6n+1$ only;

1c) A small number of trainees, after factorizing $a^2-1=(a-1)(a+1)$, are not able to correctly deduce the properties of the factors $a-1$ and $a+1$ from the hypothesis. One trainee, for example, erroneously deduces that *"a-1 and a+1 must be two odd numbers because a is non-divisible by 2"* and erroneously claims that *"$a-1=2h+1$ and $a+1=2h+3$, with $h \neq 1$ (mod3) e $h \neq 0$ (mod3) because a is not divisible by 3"*. He attaches characteristics of a to $(a-1)$ and $(a+1)$ and imposes not-justifiable conditions.

With reference to point 2, we observed that many trainees fail because:

2a) Even if they correctly set up the proof, translating, without any difficulty, the properties of a, they get stuck during the manipulation of the algebraic expressions they formulated. For example, a trainee correctly expresses a as $a=3h+1$ (h even) or as $a=3h+2$ (h odd) and, performing syntactic elaborations (substitutions and transformations), he obtains, in the first case $a^2-1=12n(3n+1)$ and in the second case $a^2-1=36n^2+24n+3$. However, he is not able to proceed in order to reach the thesis.

2b) Even when they are able to carry out syntactic elaborations, they do not always correctly interpret the new expressions they obtain. One trainee, for example, considers the case $a=6k+1$ and obtains, through syntactic transformations, the expression $(6k+1)^2-1=12k(3k+1)$, but he concludes that a^2-1 *"is divisible by 12 and not by 24"*, without considering that k or 3k+1 could be even. In this case, we can hypothesize that the standard representation of an odd number influences his erroneous conclusion.

The protocols of some trainees deserve a separate remark because they represent ruinous attempts to apply well-known proving procedures. These protocols highlight, in some cases, a total lack of understanding of the meaning of such procedures and, in other cases, serious mistakes in logic. A trainee, for example, attempting to prove the statement, proves the inverse proposition ("if a^2-1 is divisible by 24, then a is divisible by 2 and by 3"), highlighting a serious gap in logic because he thinks that proving the truth of the inverse of a proposition ensures that the original proposition is true.

FINAL CONSIDERATIONS

A high number of trainees with a degree in mathematics or physics had remarkable difficulties and gave only partial solutions, full of gaps; this fact, together with the general negative results of trainees with other scientific degrees leads us to state that: 1) those who were not appropriately educated to translate verbal expressions into the algebraic code either avoid the use of algebraic language or, in the attempt to use it to develop a proof, are unable to interpret and manage the formal expressions they formulate; 2) those who are well-trained in the syntactic manipulation of algebraic expressions get stuck if they are not appropriately educated to interpret them.

Therefore, it is essential to educate both students and trainees not only to be able to translate, but especially to interpret. It is clear that, if pre-service teachers do not master the proving activity, they are probably not able to propose it to their students in a persuasive way. Hence, we think that it is important to put the stress on proof during educational and training courses for teachers, as regards both abilities to attain a proof and the awareness of the role it plays in the mathematical activity. This "investment" is more urgent for those who are going to teach mathematics without having a background in mathematics.

In the future development of this study we want to work with teachers also on the comparison of their productions, in order to favour the construction of proofs as well as a continuous reflection about both the related difficulties and the feasibility of such activities in the classroom. In this sense, we think it is important to give them some theoretical instruments that could be a good reference in order not only to facilitate the communication between trainees and trainers, but also to make teachers aware of how to "move around in the proving universe".

NOTES

1. In Italy middle-school corresponds to grades 6^{th}, 7^{th}, 8^{th}.

2. Pre-service teachers attending SSIS (Specialisation School for secondary school teaching) at the University of Modena and Reggio Emilia.

3. By "approach" we mean "way of putting oneself in front of a problem in relation to the choice of a particular linguistic code".

REFERENCES

Alcock, L., Weber, K.: 2005, 'Referential and syntactic approaches to proof: case studies from a transition course', *Proc. PME 29*, vol.2, pp.33-40.

Arzarello, F., Bazzini, L., Chiappini, G.: 1995, The construction of algebraic knowledge: towards a socio-cultural theory and practice, Proc. *PME XIX,* Recife, vol. 1, 119-134

Barnard, T., Tall, D.: 1997, 'Cognitive units, Connections and Mathematical Proof', *Proc. PME 21*, vol.2, pp.41-48.

Boero, P., Chiappini, G., Garuti, R., Sibilla, A.: 1995, 'Towards Statements and Proof in Elementary Arithmetics', *Proc. PME 19*, vol.3, pp.129-136.

Boero, P., Douek, N., Ferrari, P.L.: 2002, 'Developing Mastery of Natural Language: Approaches to Theoretical Aspects of Mathematics', in English, L. (Ed.), *Handbook of International Research in Mathematics Education*, Mahwah (NJ, USA): Lawrence Erlbaum Associates, 241-268.

Furinghetti, F., Paola, D.: 1997, 'Shadows on proof', *Proc. PME 21*, vol.2, pp.273-280.

Friedlander A., Hershowitz R., Arcavi A.: 1989, Incipient 'algebraic' thinking in pre-algebra students, *proc. PME XIII*, vol.1, 283-290

Hanna, G.: 2000, 'Proof, explanation and exploration: An overview', *Educationl Studies in Mathematics*, 44 (1-2), 5-23.

Harel, G., Sowder, L.: 1998, 'Students' proof schemes: results from exploratory studies', In E. Dubinsky, A. Shoenfeld and J. Kaput (Eds.), *Research Issues in collegiate Mathematics Education*, *III*, American Mathematical Society, Providence, RI, pp. 234-283.

Hersh, R.: 1993, 'Proving is Convincing and Explaining', *Educational Studies in Mathematics*, 24(4) 389-399.

Hoyles, C.: 1997, 'The curricular shaping of students' approaches to Proof', *For the Learning of Mathematics*, 17, 1, 7-16.

Jones, K.: 1997, 'Student-teachers' Conceptions of Mathematical Proof', *Mathematics Education Review*, 9, 16-24.

Malara, N.A.: 2002, 'La dimostrazione in ambito aritmetico, quale spazio nella scuola secondaria?', In Malara N.A. (a cura di), *Educazione Matematica e Sviluppo Sociale: esperienze nel mondo e prospettive,* Rubettino, Soveria M. (CZ), 129-166.

Moore, R. C.: 1994, 'Making the transition to formal proof', *Educational Studies in Mathematics* 27, 249-266.

Sadovsky, P.: 1999, 'Arithmetic and algebraic practises: possible bridge between them', *Proc. PME 24*, Vol. 4, pp. 145-152.

Stylianides, A. J., Stylianides, G. J.: 2006a: 'Content knowledge for mathematics teaching: the case of reasoning and proving', *Proc. PME30*, vol. 5, pp.201-208.

Stylianides, A. J., Stylianides, G. J.: 2006b: 'Making proof central to pre-high school mathematics is an appropriate instructional goal: provable, refutable, or undecidable proposition?', *Proc. PME30*, vol. 5, pp.201-208.

Thurston, W.P.: 1994, 'On proof and progress in mathematics', *Bull American Math Society*, 30(2) 161-177.

Weber, K.: 2001, 'Student difficulty in constructing proofs: the need for strategic knowledge', *Educational Studies in Mathematics*, 48, 101-119.

Weber, K.: 2003, 'A procedural route toward understanding the concept of proof', *Proc. PME 27*, Vol. 4, pp. 395-401.

RELATIONSHIP BETWEEN BEGINNER TEACHERS IN MATHEMATICS AND THE MATHEMATICAL CONCEPT OF IMPLICATION

Virginie Deloustal-Jorrand

IUFM d'Alsace (Strasbourg, France)

& Team Maths à Modeler (Grenoble, France)

In this paper, we present a didactic analysis of the mathematical concept of implication under three points of view : sets, formal logic, deductive reasoning. For this study, our hypothesis is that most of the difficulties and mistakes, as well in the use of implication as in its understanding, are due to the lack of links in education between those three points of view. This article is in the continuation of those previously published in the acts of PME 26 and PME 28. We present here the analysis of our experimentation's results, that we have not yet shown.

INTRODUCTION

The implication is an usual object of our everyday life that we use to communicate. Its existence in natural logic leads to confuse it with the mathematical object, which then seems to be a clear object. Yet, even training teachers in mathematics have difficulties related to this concept of implication, especially with regard to necessary and sufficient conditions.

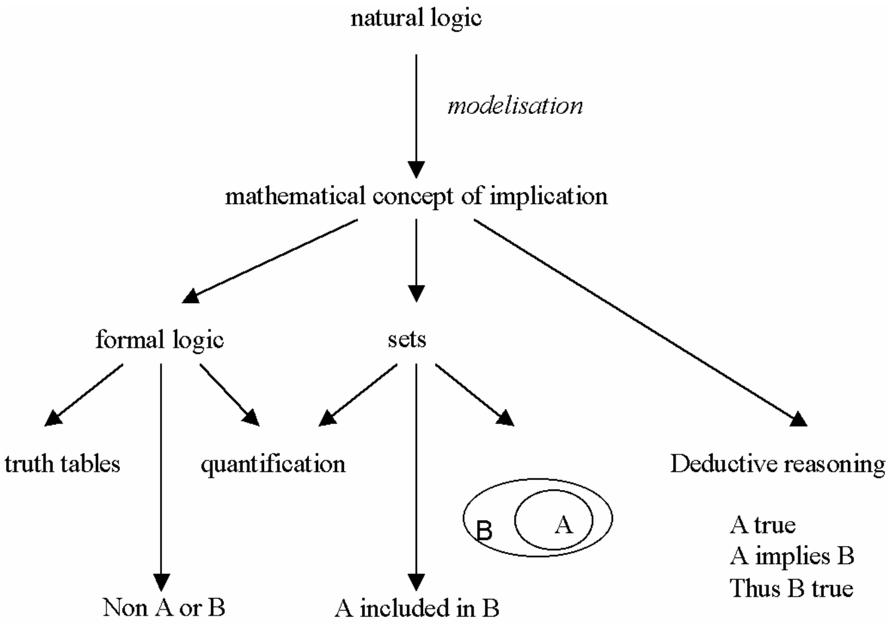
The study we present here is a part of our thesis on the mathematical concept of implication [Deloustal-Jorrand, 2004 c]. Our theorical framework is placed in the theory of french didactics, in particular, we use the tools of Vergnaud's conceptuals fields theory and those of Brousseau's didactical situations theory. Our study is linked to the work of V. Durand-Guerrier [Durand-Guerrier, 2003] on the one hand and of J. and M. Rogalski [J. & M. Rogalski, 2001] on the other hand. V. Durand-Guerrier shows, in particular, the importance of the contingent statements for the comprehension of the implication. J. and M. Rogalski try to define types of structuring of the use of logic when evaluating the truth of an implication with a false premise. We do not forget that the implication is an essential tool for the proof. Yet, we choose here to focus our research on this concept rather than on a the proof in general.

This study follows and supplements those presented at PME 26, PME 28 and ICME 10. We give now results that were lacking previously. We present, first, three points of view on the implication before a mathematical and didactical analysis of one of our problem tested with training teachers, after what we give some results and conclude.

THREE POINTS OF VIEW ON THE IMPLICATION

This paragraph was detailed in our previous research report in PME 26. Yet, we think this part of our research is necessary for the reader to understand the following problem and the aim of our research hypothesis, hence we summarize it here.

The mathematical implication seems to be a model of the natural logic implication we use in our everyday life. Like any model, this mathematical concept is faithful from certain angles to that of natural logic but not from others. This distance between the mathematical concept and the natural one leads to obstacles in the use of the mathematical concept. An epistemological analysis [Deloustal-Jorrand, 2000] enabled us to distinguish three points of view on the implication : formal logic point of view, deductive reasoning point of view, sets point of view.



Of course, these three points of view are linked and their intersections are not empty. We develop here neither the formal logic point of view (for example truth tables or formal writing of the implication) nor the "deductive reasoning" for which one can refer to Duval [Duval, 1993, p 44]. In the “deductive reasoning”, the implication object is used only as a tool. However in French secondary education, where this point of view is the only one, it often acts as a definition for the implication.

Generally speaking, having a sets point of view, means to consider that properties define sets of objects: to each property corresponds a set, the set of the objects which satisfy this property. The sets point of view on the implication can then be expressed as follows: in the set $\boldsymbol{E}$, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are respectively the set of objects satisfying the property A and the set of objects satisfying the property B. Then, the implication of B by A (i.e. A $\Rightarrow$ B) is satisfied by all the objects of the set $\boldsymbol{E}$ excluded those which are in ***A*** without being in ***B***, i.e. by all the objects located in the area shaded here after. In the particular case with $A \subset B$, all the objects in E satisfy the implication.

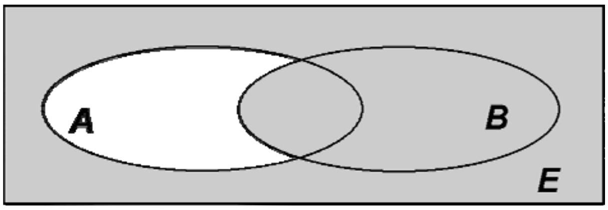


Figure 1

RESEARCH HYPOTHESIS

The experiments carried out for several years, within the framework of our research, showed that the implication was not a clear object even for beginner teachers. Moreover, they showed that, contrary to a widespread idea, a logic lecture is not enough to get rid of these mistakes and difficulties.

Following these comments, we formulated the research hypothesis: it is necessary to establish links between these three points of view on the implication for a good apprehension and a correct use of it. In this paper, we make the assumption that a *didactic engineering* [Artigue M. 1990 & 2000] linking those three points of view can be built. In the following paragraphs, we present, therefore, some of our choices for this *didactic engineering* and some of our results.

CONDITIONS OF THE *DIDACTIC ENGINEERING*

The problem we present results from an experimentation carried out in 2001 with training teachers in mathematics. We worked with two groups of approximately 25 students at the IUFM [1] of Grenoble (France). This experimentation includes two three-hour-sessions on the proof and, in particular, on the implication. The first session contained two problems (one in geometry [Deloustal-Jorrand, 2002], one on pavings [Deloustal-Jorrand, 2004 a]), the second one proposed a work on written proofs [Deloustal-Jorrand, 2004 b]. We present, in this paper, the results of the analysis of the answers to the problem of geometry (first session). Before that, we display a mathematical and didactical analysis of the first question of this problem.

MATHEMATICAL STUDY OF THE PROBLEM OF GEOMETRY

In this paragraph we show that this problem of geometry, using only easy properties, may question the reasoning in a non obvious way.

Here is the first question of our problem of geometry[2]:

> Let ABCD be a quadrilateral with two opposite sides with the same length. What conditions must diagonals satisfy to have : two other parallel sides ?

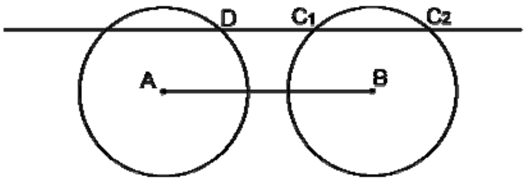
Let us call H the property "to have two opposite sides with the same length" and B the property "to have two other parallel sides". We call *H* and *B* the sets which respectively represent them. The problem is now to find A with (H and A) $\Rightarrow$ B. Let us present two approaches which may induce different solving strategies.

The first approach raises the question of sufficient conditions. First example, one may list conditions on diagonals (same length, perpendicular…) and then check if these conditions, added with the hypothesis H, imply the conclusion B. This approach puts back the problem within the deductive point of view. Second example, one can also refer to known objects. Some quadrilaterals which satisfy both H and B are well known, for example squares, rectangles, parallelograms. Besides, the properties of their diagonals are also well known, and then one can work directly with equivalences. However, if some conditions may be cheaply found, these strategies do not give the exhaustiveness, all the conditions are not *a priori* reached.

The second approach raises the question of necessary conditions. Which objects satisfy both H and B ? Then, what properties A have their diagonals ? This approach seems natural and is basically related to sets point of view. Indeed, one must consider the set $\boldsymbol{H \cap B}$. There are two ways to study those objects which satisfy H and B, either to be in ***H*** and add the property B, or to be in ***B*** and add the property H. Let us describe, in details, this first strategy, using a sets point of view.

Sets point of view strategy : H then B (H : two equal opposite sides)

Once the points A and B placed in the plane, the hypothesis (H), AD=BC, means that the points C and D belong to two same-rayed circles respectively, one centred on B, the other centred on A. Once D placed, the property (B) "two other sides parallel", means that C is the intersection of the straight line parallel with (AB) containing D with the centred on B circle. There are two intersection points C_1 and C_2.



Two configurations are thus obtained : isosceles trapezium (ABC_1D) and parallelogram (ABC_2D) [fig.1]. But one must not forget that, once A fixed, one may still change the distance AB, the ray of the circles and the position of D (linked to that of C) on its circle. So, when the two circles intersect, there is a new configuration : a cross quadrilateral called CQ_1 (ABC_1D) [fig.2]

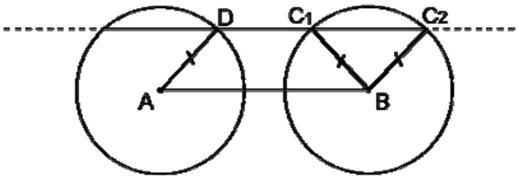


Fig.1

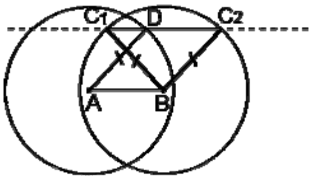
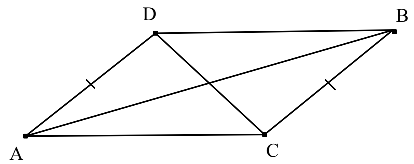


Fig.2

So there is the implication : (H and B) $\Rightarrow$ (parallelogram or isosceles trapezium or cross quadrilateral CQ_1). We thus know the configurations which satisfy both H and B, it remains then to find the conditions on the diagonals.

However, for a quadrilateral, being a parallelogram is equivalent to having diagonals which cross in their middle. Isosceles trapeziums and cross quadrilaterals CQ_1 have same-lengthed diagonals. Now remains to see whether "to have same-lengthed diagonals" (A_1) is a sufficient condition, i.e. if the implication in the quadrilaterals: (H) and (A_1) $\Rightarrow$ (isosceles trapezium or cross quadrilateral CQ_1) is true.

For that, the sets point of view is necessary again, we have to study the quadrilaterals which satisfy (H) and (A_1).We will not detail the rest of the solving, but let us say that these two conditions bring obviously the isosceles trapezium and the cross quadrilateral CQ_1 but also a cross quadrilateral CQ_2 (cf. here below) which does not satisfy the conclusion (B). The condition "to have same-lengthed diagonals" is thus not sufficient and will have to be restricted to exclude CQ_2. The final solving of this exercise is not the subject of this article, but we wanted to show how this problem can question the implication.



Cross parallelogramm CQ_2

DIDACTICAL STUDY

We present, first, some general choices for our *didactic engineering*, then the choices concerning, more precisely, this problem of geometry. Let us call again H the property “to have two opposite sides with the same length” and B the property “to have two other parallel sides”, respectively *H* and *B* the corresponding sets.

Mathematical framework for our *didactic engineering*

First of all, we choose, for our experimentations, very easily accessible mathematical concepts. Indeed, our hypothesis is that, to see a work on the reasoning and distinguish difficulties due to the concept of implication, there must not be difficulties linked to a mathematical concept. This problem contains only notions very well known by students such as quadrilaterals, parallelograms, diagonals...

Real question

Our hypothesis is that the question must be difficult enough to allow a work on the reasoning. Besides, the truth of the implication must be questioned. Thus, we ban questions like: “Prove that A $\Rightarrow$ B is true”.

Implication versus equivalence

A problem whith equivalences ($\Leftrightarrow$) does not allow a work on the implication. We choose for our experimentation to distinguish a necessary and a sufficient condition.

Practical organization of the sessions

Our hypothesis is that a research in groups is necessary for our *didactic engineering*. That allows a confrontation between the various points of view. Nevertheless, the first individual work gives each one the time to make his own opinion about the problems. These various opinions will feed the discussions.

Choices to put forward the three points of view in our *didactic engineering*

We made the assumption that the “deductive reasoning” is usually always present. Therefore, in our *didactic engineering,* we chose to emphasize the sets point of view and the logical one, depending on the problems.

Choices to put forward the sets point of view in this geometry problem

The objects questioned in this problem are not often considered in the french secondary school.

First, H is the set of the quadrilaterals having two equal opposite sides. Their properties are not as well known as those of parallelogramms for example. Yet, in french secondary school, one usually considers, in fact, the implication "in H, $A \Rightarrow B$". Here H is implicit because very well known and used. For example, most of the time in the parallelograms’ class, properties are implicitly used (for example, convexity). Here, the property H must be explicit during all the resolution.

Besides, H, A and B contain crossed quadrilaterals which are not teached in french secondary school. The training teachers must distinguish the crossed quadrilaterals and define them.

Therefore, the presence of these quadrilaterals should push the strategies linked to the sets point of view. Indeed, the training teachers’ usual knowledge on the quadrilaterals is not sufficient to give a correct answer.

Choices for a work on the implication in this geometry problem

The condition A is unknown. Hence, the deductive reasoning is “upended” here.

It is hard to find A by using the reasoning: “in H, $\neg B \Rightarrow \neg A$”. Indeed, it is difficult to define the set corresponding to $\neg B$ “Not to have two other parallel sides”.

The condition A sought is, in fact, equivalent to (A_1 or A_2). Usually, the taught implications are like (H **and** A1 **and** A2) $\Rightarrow$ B, whereas, here, it is (H and [A1 **or** A2]) $\Rightarrow$ B. Consequently, we think A is more difficult to define and prove.

At last, the problem itself forces to question the direction of the implication ($\Rightarrow$, $\Leftarrow$). Indeed, we did not specify if the requested conditions were necessary or sufficient.

Paper Vs. interactive geometry software in this geometry problem

A part of the task consists in varying the parameters of the figure (length of the ray, position of D on the circle…). An interactive geometry software (cabri for example) would make this part of the resolution easier. However, we assume that the use of this software would prevent us to check whether the sets point of view is used or not. Indeed, we would not be able to see if the training teachers are working on the sketch itself (as seen on the screen) or on the figure (as a representation of the sets). In fact, with the software, the manipulation of the sketch is mainly due to the "didactical contract". That is to say, the user has no choice but to manipulate the sketch. Consequently, this manipulation is not, most of the time, the result of a "mathematical choice" and is not a clue of the presence of the sets point of view. Hence, we chose to work on paper rather than with a software.

RESULTS

Working on the implication

First of all, we can say there was a real mathematical question in this problem. Although students first thought this exercise very easy, its solution required a very long research in groups. The answers are incomplete and, in the end, the students declared this exercise very difficult.

Robert: It is an exercise which as a teacher, I would not give before university

On one hand, no group found the problem obvious but, on the other hand, no group was stopped by mathematical difficulties. They had strategies to begin their research. That is why we can assert that a work on reasoning and implication was done.

The exercise fulfilled its role, as for the work on the implication. Indeed discussions about necessary and sufficient conditions took place in the groups during all the research. The teachers looked, first, for sufficient conditions but, because of the problem, they had to look for necessary conditions too.

Anne: We looked for sufficient but…if we consider it, maybe…it is necessary too.

Thus, there were a lot of questions in all the groups. For example, they all wondered if the property "to have same-lengthed diagonals" was a sufficient condition. We saw then teachers confusing sufficient condition and necessary condition.

Antoine: No, « same-lengthed diagonals » is not sufficient. A parallelogramm is OK, yet its diagonals don't have the same length.

This counter-example is not valid, it shows that the condition is not necessary but it does not show anything on the sufficient quality.

Besides, all the groups were looking for a "complete" condition merging all the possible conditions. They all found the sufficient condition "to have diagonals which cross in the middle" but they all thought it was not enough to answer the problem.

Armelle: I was looking for a condition that would meet…all possibilities.

Moreover, most of the groups tried to define a "minimal" condition, which requires the fewest hypotheses. Then, they had to take the property H into account.

Anne: Until now, we didn't use AB=CD ! Thus if we take into account this new condition, maybe we can formulate more easily our condition on the diagonals.

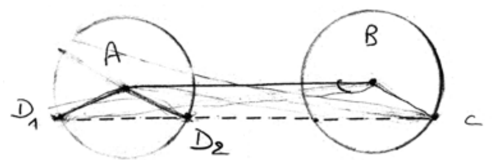
Furthermore, as we had planned, they were not able to use the reasoning "in H, $\neg B \Rightarrow \neg A$" in order to find A. But one group tried and had to argue about it.

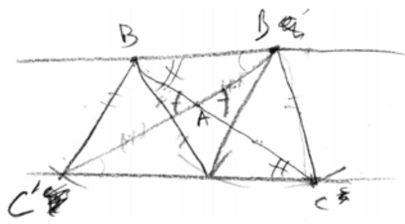
Davy: But in fact…no, the problem is, what are you going to look for on your diagonals ? What are you going to deny on your diagonals ?

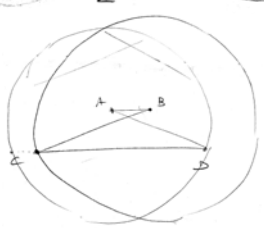
Working on the sets point of view

Most of the groups did not know *how* to solve this problem. This confirms our presumption that the ensemblist point of view is necessary in this situation. The analyse of the answers shows that the sets point of view is not an available tool for the training teachers. But on the other hand, we can see a lot of implicit clues of its presence. We now want to describe these clues.

First of all, all the groups used sketches in their strategies. Some of them are similar to those of the strategy based on sets point of view in the mathematical analysis.







Nevertheless, they did not check the variation of their sketches (length of AB, BC…). As a result, they did not check that their sketches represented all the possible quadrilaterals. Thus, they can not be sure that they found all the possibilities for ABCD. Yet, these sketches show that the quadrilaterals were built as sets of points.

Besides, the conjectures, examples, counter-examples are very present. These are marks of the sets point of view since teachers have to speak about sets.

Furthermore, the teachers often forgot the property H "two same-lengthed sides", especially when they seeked counter-examples.

Laura : But, your opposite sides with the same length, where are they here ?

Antoine : Oh yes, I made a mistake.

Yet, before long, the situation itself and the work in groups allowed them to see that the property H was not taken into account. No group kept a false counter-example. This property fulfilled its role since the teachers had to remind it during the resolution and had to consider the set of quadrilaterals which satisfy H.

Despite it, the sets point of view is not a tool for the teachers as we show now.

All the groups left apart the crossed quadrilaterals, explicitly or implicitly when they began their resolution. Some groups took them into account afterwards, but they separated this new research. They gave some reasons: crossed quadrilaterals are not interesting, they are particular objects, they do not have diagonals…

Robert: But why do you draw a crossed one, it is not an interesting case!

Carine: You can't talk about diagonals if you link this with this. Thus necessarily it is not crossed.

Besides, the sketches are not seen as a tool of the set point of view. Then they can not be considered as a proof. Yet, all the teachers did use sketches. That is why, one of the main questions during the resolution, in all the groups, was to find what could be the role of these sketches. Do they give all the possible quadrilaterals ? If there are enough sketches can one be sure to have all the quadrilaterals ? Can they be considered as a proof ?

Laura: To prove is not to draw !

For the training teachers, a sketch can not be a proof whereas this can be true in the sets point of view as we showed in the mathematical analysis.

Working on the logical point of view

Most of the groups gave different sufficient conditions. Only one group gave a single condition written with the logical word "**or**". Generally, the equivalence $[A_1 \Rightarrow P$ and $A_2 \Rightarrow P] \Leftrightarrow [(A_1$ or $A_2) \Rightarrow P]$ is not admitted, even when proposed by a teacher. This situation allows a discussion on this equivalence.

CONCLUSION

We have shown that this problem allows to question the implication and the proof. Indeed, the training teachers have difficulties which are not related to mathematical objects. They had to examine the conditions to know wether they were sufficient or necessary. They discussed about what a mathematical proof is.

Moreover this problem requires a work under the sets point of view. Although it is not used as a tool, it appears many times, concerning counter-examples or the role of sketches. Besides, the logical point of view appears too, especially to express the final condition with the "or", but also to find a minimal condition or sometimes to decide if a condition is sufficient or necessary.

Lastly, these results are to be placed among others. Indeed, this problem of geometry forms part of a six hour experimentation including other stages of work, in particular, studies, in groups, of written proofs [Deloustal-Jorrand, 2004 b] and of a problem of discrete mathematics [Deloustal-Jorrand, 2004 a]. Moreover, this experimentation takes sense when one knows that it was preceded by two others, carried out in 1999 and 2000.This problem of geometry is, thus, to consider as part of a broader context.

NOTES

1. University Institute for Teacher Training

2. There were two other questions: two 90 degrees angle ? ; two other same-lengthed sides ?.

REFERENCES

Artigue, M. (1990) Ingénierie didactique, *Recherches en Didactique des Mathématiques Vol.9, n°3*, 281-308.

Artigue, M. (2000) Didactic engineering and the complexity of learnig processes in classroom situations, *In C. Bergsten, G. Dahland, B. Grevholm (eds), Proccedings of the MADIF 2 Conference, Gothengurg, Swedish Society for Research in Mathematics Education*, 5-20.

Brousseau, G. (1997) Theory of Didactical Situations in Mathematics (translated and edited by N. Balacheff, M. Cooper, R. Sutherland, V. Warfield), *Kluwer Academic Publishers*.

Deloustal-Jorrand, V. (2000) L'implication. Quelques aspects dans les manuels et points de vue d'élèves-professeurs, *Petit x,* 55.

Deloustal-Jorrand, V. (2002) Implication and mathematical reasoning, *26th Conference for the Psychology of Mathematics Education, Norwich.*

Deloustal-Jorrand, V. (2004 a) Polyminos : a way to teach the mathematical concept of implication, *X° ICME Congress, Copenhagen,* electronic publication: http://www.icme-organisers.dk/tsg19/Deloustal.htm

Deloustal-Jorrand, V. (2004 b) Studying the mathematical concept of implication through a problem on written proofs, *28th Conference for the Psychology of Mathematics Education, Bergen.*

Deloustal-Jorrand, V. (2004 c) L'implication mathématique: étude épistémologique et didactique, *Thesis, University Joseph-Fourier, Grenoble, France.*

Douady, R. (1985) The interplay between different settings, *9th Conference for the Psychology of Mathematics Education, Noordwijkerhout*.

Durand-Guerrier, V. (2003) Which notion of implication is the right one ? From logical considerations to a didactic perspective, *Educational Studies in Mathematics 53*, 5-34.

Durand-Guerrier, V. (2005) Natural deduction in Predicate Calculus. A tool for analysing proof in a didactic perspective, *CERME-4 Congress, Spain*, electronic publication: http://www.lettredelapreuve.it/Newsletter/05Automne/CERME4Durand.pdf

Duval, R. (1993) Argumenter, démontrer, expliquer : continuité ou rupture cognitive ?, *Petit x,* 31.

Rogalski, J. and Rogalski, M. (2001) How do graduate mathematics students evaluate assertions with a false premise ?, *25th Conference for the Psychology of Mathematics Education, Utrecht.*

Vergnaud, G. (1988) Theoretical Frameworks and Emperical Facts in the Psychology of Mathematics Education, *VI° ICME Congress*, Hungary.

USING THE VAN HIELE THEORY TO ANALYSE THE TEACHING OF GEOMETRICAL PROOF AT GRADE 8 IN SHANGHAI

Liping Ding and Keith Jones

University of Southampton, U.K.

The data reported in this paper come from a study aimed at explaining how successful teachers teach proof in geometry. Through a careful analysis of a series of lessons taught in Grade 8 in Shanghai, China, the paper reports on the appropriateness of the van Hiele model of 'teaching phases' within the Chinese context. The analysis indicates that though the second and third van Hiele teaching phases could be identified in the Chinese lessons, the instructional complexity of, for example, the guided orientation phase means that more research is needed into the validity of the van Hiele model of teaching.

INTRODUCTION

The teaching of geometry, and, in particular, the teaching of geometrical proof, has received changing amounts of emphasis in recent curriculum reforms across many countries (compare, for example, the US NCTM *Standards*, 1989, 2000). For many, such as Wu (1996), plane geometry, taught well, is essential as it can give students at secondary school a first experience of the power and the economy of the basic axiom-theorem-deductive feature of mathematics. In China, the process and method of proof continues to be considered as an essential part of the school mathematics curriculum. For example, the *Shanghai Primary and Secondary School Curriculum Standard* (Shanghai Education Committee, 2004) specifies, for the lower secondary school level (Grade 6 to Grade 9; students' age 11-15 years), that the process of proving should be emphasized for the following reasons:

> ...to help students experience the developmental process from intuitive geometry to experimental geometry and then to deductive geometry; to establish the relationship and recognize the distinction between intuition and logical thinking; to perceive the meaning and the use of inductive reasoning, analogical reasoning, and deductive reasoning…; to experience the process of 'experiment-induction-conjecture-proof' (p35, translated by Ding).

Given the continuing debate across the world about the learning and teaching proof in geometry and the difficulties that many students encounter with this topic (see, for example, Jones 2000; Mammana and Villani, 1998), the research from which this paper is taken aims to contribute to understanding and interpreting, in depth, the teaching of geometrical proof by analysing classroom instruction at Grade 8 in Shanghai, China. The aim of this paper, following Whitman *et al* (1997), is to analyse the appropriateness of the van Hiele model of 'teaching phases' (see below) within the Chinese context, and, in particular, to see how well the model characterises the observed teaching in order to try to *explain* how a successful teacher teaches what is, by all accounts, an aspect of mathematics that is very difficult for many students at school.

RESEARCH VIEWS ON THE VAN HIELE THEORY

Based on their pedagogical experience and their teaching experiments, the van Hieles (husband and wife) proposed a psychological/pedagogical theory of thought levels in geometry (English version in Geddes *et al*., 1984). For many researchers, such as Schoenfeld (1986), this model of thought levels provides a useful empirically-based description of what are likely to be relatively stable, qualitatively different, states or levels of understanding in learners. Accompanying this model of thought levels, the van Hieles proposed a model of teaching that specifies five sequential phases of instruction (see, for example, Clements & Battista, 1992, pp430-1) that, the van Hieles suggest, are a means of enhancing students' thinking from one thought level to the next. This model of teaching phases, as discussed below, is used as the main theoretical framework for this paper.

Originally, and in an attempt to understand the structure of geometry learning, Dina van Hiele-Geldof (see Geddes *et al*, 1984, pp217-223) focused on analyzing the relationship between student and subject matter in elementary geometry. As a result of her research, she suggested five teaching phases which, for the purposes of this paper, are termed as follows: 1) *Information*; 2) *Guided Orientation*; 3) *Explicitation*; 4) *Free Orientation*; 5) *Integration* (adapted from Clements & Battista, 1992, pp430-1; Geddes *et al*, 1984, p223; Hoffer, 1983).

At this point it is worth noting Hoffer's (1983) view that the third phase (*Explicitation*) was incorrectly given by Wirszup (1976, p83) as 'explanation', with Hoffer taking the view that, in this third phase, it is essential that "students make the observations explicitly rather than receive lectures (explanations) from the teacher" (*op cit*, p208). Furthermore, Clements and Battista (1992, pp430-1) call the second phase *Guided Orientation*, rather than use the Geddes *et al* term *Direct Orientation*.

Whatever the terms used, and the above illustrated some of the unresolved issues about the choice of terminology, the model is quite loose in that, as Schoenfeld (1986, p252) explains, and as Whitman *et al* (1997) found, the nature of the pedagogical sequence is far from clear. Not only that, but as the model is more a suggested process than a fixed formula, it is not at all obvious whether it is necessary for the teacher to go through each and every phase. Indeed, Hershkowitz (1998) is of the view that the van Hiele theory does not account well for the relationship between the context of the learning environment and the mathematical reasoning being developed. She suggests more context-specific research and this matches the call by Whitman *et al* (*ibid* p217) for more research to evaluate the use of the van Hiele theory with students of different cultural backgrounds. In general, the existing van Hiele-based research has yet to address systematically any of these issues concerning the nature and specification of the teaching phases.

In the little research that has directly examined the van Hiele teaching phases, Hoffer (1994) developed a way of codifying teacher behaviour in terms of the phases of instruction (which he characterised as "Familiarization", "Guided Orientation", "Free Orientation", "Verbalization", "Integration"). He then tested the coding procedure on a number of mathematics classes. Amongst his findings were that US mathematics teachers (not familiar with the van Hiele teaching phases) demonstrated a preponderance of phase 2 instruction (that is, "Guided Orientation") and, Hoffer claims, often interrupted student progress toward higher levels in order to return to phase 2 instruction. Taking up the Hoffer approach, Whitman *et al* (1997) applied Hoffer's instrument to the comparative study of geometry instruction in Japan and the US. What they found was that the US teacher, in general, taught using phase 2 instruction (that is, "Guided Orientation") but that "the class showed multiple phases ….within one module" (*ibid* p228) whereas in the case of the Japanese teacher "there was ambiguity in trying to identify the phase at which the teacher was teaching because it appeared that more than one interpretation was available [to the research team]" (*ibid* p229). In both these cases, while Hoffer studied a number of teachers, and while Whitman *et al* selected lessons on congruence of triangles from one Japanese and one US teacher, the actual subject matter being taught received little attention in their published papers.

To contribute to the research base for this aspect of the van Hiele theory, and following Whitman *et al* (1997), the data reported in this paper come from a study aimed at seeing how well the van Hiele model of the five teaching phases accounts for the pedagogical methods used in teaching deductive geometry in classrooms in China. The key research question being addressed is to what extent the van Hiele model of five teaching phases accounts for the teaching of geometric proof by successful teachers in Chinese classrooms.

METHODOLOGICAL CONSIDERATIONS

The data reported in the paper come from a study of geometry teaching at Grade 8 in Shanghai (for other details, see Ding & Jones, 2006). In the city there are four grades at the lower secondary school level, from Grade 6 (students' age, 11-12 years old) to Grade 9 (students' age, 14-15 years old). As the school geometry curriculum is divided into three stages, namely intuitive, experimental and deductive geometry, students at Grade 8 (13-14 years old) start to learn more formal deductive geometry and practice proof writing. Consequently, studying this Grade offers the opportunity to analyse how Chinese teachers lead students at this Grade level to learn proof in deductive geometry.

For the purposes of this paper, data, collected in 2006, is selected from the teaching of one teacher, referred to as Lily (pseudonym), in an ordinary public school in a typical suburb of the city. The teacher, selected because of very good reputation for student success, had over 20 years teaching experience of secondary school mathematics. At the time of the data collection, there were 39 students in the class and mathematics lessons, each 40 minutes long, took place six times each week. Every lesson with this teacher was observed over a three week period. During this time, 12 geometry lessons were observed with topics concerning parallelograms, rectangles, rhombi and squares. In total, four definitions and fifteen theorems were taught during the three-week observation period. Given the known expertise of the teacher, supporting evidence showed that the students were ready for this level of mathematics.

The data collected included classroom observations notes, audio-recordings of lessons (transcribed), and other field notes. During each lesson, photographs were taken to provide information which could not be recorded by audio-recorder or field notes (for example, recording work presented on the blackboard).

USING THE MODEL OF TEACHING PHASES TO ANALYSE LESSONS

In analysing the data, it was vital to understand, in depth, the nature of each phase in the van Hiele model. Pierre van Hiele (1986, p177) suggested that the teacher conducts the teaching process as follows: in the first phase, "by placing at the children's disposal (putting into discussion) material clarifying the context"; in the second phase, "by supplying the material by which the pupils learn the principal connections in the field of thinking"; in the third phase, "by leading class discussions that will end in a correct use of language"; in the fourth phase, "by supplying materials with various possibilities of use and giving instructions to permit various performances"; in the fifth phase, "by inviting the pupils to reflect on their actions, by having rules composed and memorized, and so on". This illustrates that, as a teacher moves through the teaching phases, there is a transition from forms of direct instruction towards the students' independence from the teacher.

After a very careful study of the van Hieles' original work, together with van Hiele-based research on the teaching phases, we seek to formulate an operational characterisation of the teaching phases in geometrical proof teaching and use this to analyse data collected in the Chinese classroom. The characteristics and terms of each phase described by the van Hieles (see Geddes *et al*., 1984), Hoffer (1983, 1994) and Clements and Battista (1992) were utilised. In what follows, an analysis of the teaching of proof in two geometry lessons (lesson Z2 and lesson Z3 - designations for identification purposes only) given by the case-study teacher, Lily (pseudonym), is presented in which each of the van Hiele phases is practically characterised. In these two lessons, there were two types of proof teaching: 1) teaching new geometrical theorems (Proof 1 and 2, involving theorems verifying a parallelogram by its opposite sides); 2) teaching proof of problem solving, namely, exercises consisting of two relatively simple problems (Exercises 4-5) and three complex problems (Exercises 6-8).

Characterizing the *Information* phase of teaching

The *Information* phase can be characterised when the teacher provides inquiry-based learning activities in which students carry out 'experiments' and make inductive reasoning and conjectures relating to a geometrical proof. In the analysis of the observed lessons, this phase was not found in either lesson, perhaps because the observed lessons were not at the start of the teaching of geometrical proof to these particular students.

Characterizing the *Guided Orientation* phase of teaching

In the analysed lessons, the phase of *Guided Orientation* was characterised by the teacher guiding students to uncover the links that form relationships of a proof problem, as exemplified by the following extract related to Proof 3 of lesson Z2 (see Figure 2)

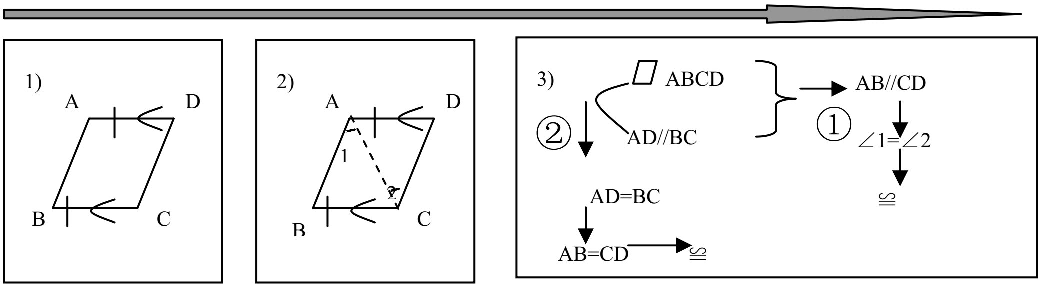


Figure 1: Proof 3, lesson Z2

The teacher briefly presented the ‘given’ for the problem (AD//=BC) and the statement to be proved (ABCD is a parallelogram), putting marks for the ‘given’ on the figure on the blackboard (see figure 1-1).

91 Lily: So far, ***how many methods did we learn to verify a parallelogram?*** (Some students answered the definition (AB//CD, AD//BC), and some answered Proof2 (from the previous proof, students know that AB=CD, AD=BC); detailed student dialogue omitted)

101 Lily: OK. Now, if I need to prove that this is a parallelogram, ***what is given?*** (Some students suggested AD//BC, some talked about AD=BC; detailed student dialogue omitted)

107 Lily: ***How do you make a decision?*** (Some students suggested the definition (AB//CD, AD//BC), while others suggested AB=CD, AD=BC; the teacher highlighted the given AD//BC, students discussed the use of the definition; detailed student dialogue omitted)

115 Lily: If I use the definition to prove, ***what should I prove first?***

116 Linlin (Boy): Parallel sides.

120 Lily: ***How to prove the parallel lines?*** (AB//CD). (Students suggested linking AC; the teacher used a board ruler to link AC - see figure 1-2; student dialogue omitted)

126 Lily: To prove AB//CD, ***what should I turn to prove first***? (Some students discussed equal angles, some answered alternate interior angles; student dialogue omitted)

129 Lily: ***Which pair of angles?*** (Using the students’ answers, the teacher highlight angles BAC and ACD; see figure 1-2; detailed student dialogue omitted.)

132 Lily: To prove $\angle 1 = \angle 2$, ***what should we turn to prove first…?*** (The class then discussed the idea of proving congruent triangles; dialogue omitted)

(While the teacher asked students these questions, she gradually wrote down an analytic structure of the proof on the blackboard - see figure 1-3 . She then used a similar sequence of questions to organize the analytic structure of another proof; see figure 1-3②)

Characterizing the *Explicitation* phase of teaching

The *Explicitation* phase of teaching was determined when students had knowledge, and were able to use mathematical language, to present the general structure of a proof. For instance, the extract from Exercise 4 of lesson Z2 (see figure 2) is characteristic of the explicitation phase. The extract follows the teacher explaining that ABCD (figure 2-1) is a parallelogram and that points E and F are 'dynamic' points that can move such that BE is always equal to DF (figure 2-2). The problem to prove what shape is quadrilateral BEDF (figure 2-3).

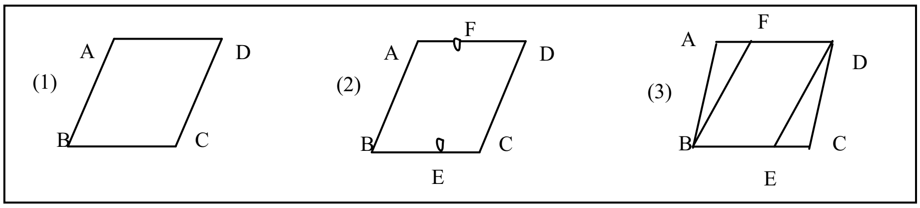


Figure 2: Exercise 4, lesson Z2

210 Lily: ***What does quadrilateral BEDF look like?*** (Students answer a parallelogram, dialogue omitted; the teacher asks the student to discuss why this might be the case)

206.1 Beibei: If a pair of opposite sides is equal and parallel, then….

209 Liuliu: (responded to Beibei) Yes, parallel and equal…???

215 Liuliu: Opposite sides are equal; I could use this to prove this problem. (this statement is taken to mean FD=BE, BF=DE).

221.1 Beibei: (Responding on Liuliu) Why?

221.2 Liuliu: You could see here. First, to calculate that ABF and ECD are congruent. Next, BF and DE are congruent. Oh, equal. BE and FD are already known.

221.3 Liuliu: This is to prove quadrilateral BEDF is a parallelogram.

221.4 Beibei: It is already given that a pair of opposite sides is equal.

221.5 Liuliu: You need to calculate that its opposite sides are equal. One pair of sides is given, yet you need to know another pair of sides.

221.6 Beibei: It is already given that BE=FD.

221.7 Liuliu: BE=DF. But you need to prove that BF=DE.

221.8 Beibei: If a pair of opposite sides of a quadrilateral is not only equal, but also parallel, then it is a parallelogram. (Liuliu does not reply to Beibei at this point; both listen to another student's presentation of the proof invited by the teacher.)

Characterizing the *Free orientation* phase of teaching

The *Free Orientation* phase of teaching, according to the van Hiele model and in the context of teaching geometrical proof, is when students learn their own ways to prove multi-step proof problems. This phase was not found in Lily's lesson 2 and 3, perhaps because the sampled lessons were in the *Guided Orientation* phase of teaching.

Characterizing the *Integration* phase of teaching

The *Integration* phase of teaching, according to the van Hiele model and in the context of teaching geometrical proof, is when students review and reflect the methods used in a set of proofs. This phase was not found in Lily's lesson 2 and 3, perhaps because the sampled lessons were in the *Guided Orientation* phase of teaching.

DEVELOPING AN OPERATIONAL MODEL OF THE VAN HIELE PHASES

An operational model of the van Hiele phases for the *process of teaching proof in geometry* is proposed as one outcome of this analysis. Descriptors of the *Guided Orientation* phase of this framework were drawn from a detailed analysis, exemplified above, of the case study lessons. The operational model is arranged in terms of the van Hiele phases of teaching:

1. <u>Information</u>: The teacher provides students inquiry-based learning activities in which students do experiments and make inductive reasoning and conjecture for a proof.

2. <u>Guided Orientation</u>: The teacher guides students to uncover the links that form a proof.

--a) The teacher demonstrates the 'Given' and the 'To Prove' statement or a problem; draws a figure and put marks on the figure on the blackboard; asks a set of questions and corrects students' answers to help them understand the requirement of a problem; provides students time to read the problem and to draw the figure on their own.

--b) The teacher encourages students to outline the different known theorems of a figure; helps students review the nature of the known definition/theorem and uncover their relationship; guides students to use deductive method to obtain new theorem from other known definition/theorems; shows how to write a formal proof; helps students evaluate the nature of the new theorem; guides students to use words and mathematical language to precisely present the new theorem.

--c) The teacher encourages students to outline the different ways to prove a problem; guides students to present the general structure of a proof and correct errors and emphasizes the rigor in proving; demonstrates the use of a new theorem in solving a set of problems.

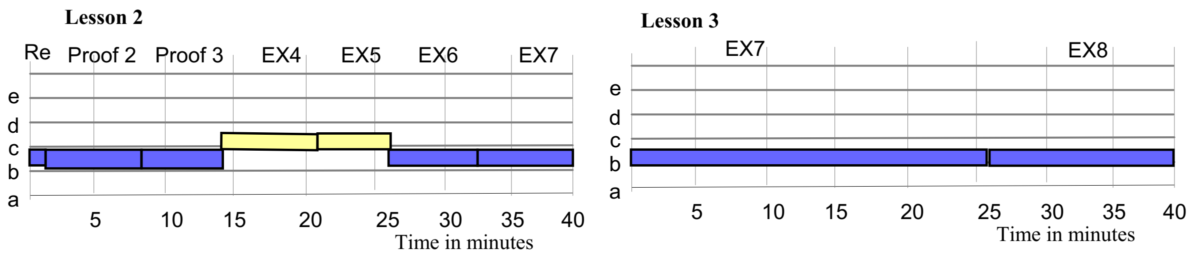
--d) The teacher provides multi-step problems that help students understand the network of definition/theorems; encourages students discover the hidden property by a set of questions and by uncovering a basic figure from the complicated figure; guides student to evaluate an appreciate method of a proof; helps students recognize the nature of different theorems of a figure;

3. Explicitation: The teacher ensures that students have the knowledge to present ideas and the general structure of a proof before the teacher's guidance. S/he begins to accurately use mathematical language in presenting a proof. In this phase, the teacher gets to understand what students have learned of the proof topic.

4. Free Orientation: The teacher ensures that students learn their own way to prove multi-step problems, often in a variety of ways.

5. Integration: The teacher ensures that students review, and reflect on, the methods used in a set of proofs.

Using this operational model, the teaching of proof in Lily's lesson 2 and 3 is shown in Figure 3.



a- Information; b- Guided Orientation; c- Explicitation; d- Free Orientation; e- Integration. EX-exercises

[colours not significant]

Figure 3: Proof teaching phases in Lily's lesson Z2 and Z3

DISCUSSION

The analysis presented in this paper indicates that the van Hiele theory can be a way of characterising the teaching phases in geometrical proof. In studying the relevant research, and in carrying out the analysis presented in this paper, it is clear that many questions about the teaching phases remain unanswered. As Clements and Battista (1992, p434) note, overall, and primarily because of a lack of research, many issues remains unclear, including how the phases of teaching relate to the subject matter and the students' prior attainment, whether the phases are followed in a linear fashion or iteratively within topic or even within individual lessons, whether one or more mathematical concepts can be included within one sequence of teaching phases, whether a different emphasis on particular phases depends on what is being taught (such as concepts, or skills, or problem-solving), and so on.

In terms of how long a teaching phase may last, Hoffer (1994), in his study, broken down lessons into discernible activities lasting 3-20 minutes and codified these in terms of the van Hiele teaching phases. In analysing the geometry lessons observed in Shanghai, the second and third of the van Hiele teaching phases were found across the range of lessons observed for this project (beyond the two lessons reported in this paper). Even so, the study indicates that the instructional complexity of the 'guided orientation' phase means that far more research is needed in the van Hiele teaching phases. For example, in lesson Z2 and Z3 (as analysed in this paper), the teacher's intention was carefully to lead students to experience the systematic network of theorems in constructing a proof through a sequence of well-designed, though demanding, multi-steps exercises. Moreover, the analysis of the instructional structure of the individual problem in the lesson suggests that the teacher was likely to develop students' abstract thinking and extend the structure of thinking through the model 'new theorem - simple problems - complicated problems'. According to interviews conducted with the teacher, she considered mathematical problems as a means of helping students practice the use of new theorems in further proofs. In terms of her instructional view, there were two types of problems in proof teaching: 1) simple problem, by which she meant one-step problems which directly use the new theorem; 2) complicated problem, which, for her, consist of both 'latitudinal' and 'longitudinal' problems – with a latitudinal problem containing a system of knowledge, (for instance, theorems of a parallelogram may link to those of a triangle or a circle, a parallelogram may link to function or equation) and a longitudinal problem entailing using a theorem in depth in a proof (for instance, using a theorem twice in a proof, with the second use probably requiring the use of an auxiliary line).

All these considerations means that further study is essential if *explanations* of how teachers, in China or elsewhere, effectively support students to extend their geometric thinking and proving. Given the aim of this study is interpreting, in depth, the teaching of geometrical proof in classroom, the intention is that the operational model of the van Hiele phases proposed in this paper (based primarily on two case study lessons) is to be further refined through additional analysis of all observed data.

ACKNOWLEDGEMENTS: the authors would like to record their appreciation to ERME members and colleagues in China for helpful comments on earlier drafts of this paper.

REFERENCES

Clements, D. H. and Battista, M. T.: 1992, Geometry and spatial reasoning. In D.A. Grouws (ed.), *Handbook of Research on Mathematics Teaching and Learning*. New York: Macmillan.

Ding, L. & Jones, K.: 2006, Teaching geometry in lower secondary school in Shanghai, China, *Proceedings of the British Society for Research into Learning Mathematics*, 26(1), 41-46.

Geddes, D., Fuys, D. and Tischler, R. (Eds.): 1984, *English Translation of Selected Writings of Dina van Hiele-Geldof and Pierre M. van Hiele.* Washington, D.C.: Research in Science Education (RISE) Program of the NSF.

Hershkowitz, R.: 1998, Organization and freedom in geometry learning and teaching. In R. Lehrer, and D. Chazan (eds.) *Designing Learning Environments for Developing Understanding of Geometry and Space*. Mahwah, NJ: LEA.

Hoffer, A.: 1983, van Hiele-Based Research. In R. Lesh and M. Landau (eds.) *Acquisition of Mathematics Concepts and Processes.* Orlando, Fla: Academic Press.

Hoffer, A.: 1994, *An Instrument to Evaluate Classroom Interaction and Communication in Mathematics.* Irvine, CA: the author.

Jones, K.: 2000, Critical issues in the design of the geometry curriculum. *In*: B. Barton (Ed), *Readings in Mathematics Education*. Auckland, NZ: University of Auckland. pp 75-90.

Mammana, C. and Villani, V. (eds.): 1998, *Perspectives on the Teaching of Geometry for the 21st Century, an ICMI Study*. Dordrecht: Kluwer

National Council of Teachers of Mathematics: 1989, *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics: 2000, *Principles and standards for school mathematics.* Reston, VA: NCTM.

Schoenfeld, A. H.: 1986, On having and using geometric knowledge. In J. Hiebert (ed.) C*onceptual and Procedural Knowledge: the case of mathematics.* (pp. 225-264). Hillsdale, NJ: LEA.

Shanghai Education Committee: 2004, *Shanghai Primary and Secondary School Curriculum Standard*. Shanghai: Shanghai Education Press. (上海市教育委员, 上海市中小学数学课程标准, 试行稿. 上海教育出版社).

Van Hiele, P.M.: 1986, *Structure and insight.* New York: Academic Press.

Whitman, N. C., Nohda, N., Lai, M. K., Hashimoto, Y., Iijima, Y., Isoda, M. and Hoffer, A.: 1997, Mathematics education: a cross-cultural study, *Peabody Journal of Education,* 72(1), 215-232.

Wirszup, I.: 1976, Breakthrough in the psychology of learning and teaching geometry. In J. L. Martin and D. A. Bradbard (eds.) *Space and Geometry: Papers from a research workshop*, pp.75-97. Athens, GA: University of Georgia.

Wu, H.: 1996, The role of Euclidean geometry in high school, *Journal of Mathematical Behavior,* 15(3), 221-237.

ANALYSIS OF CONJECTURES AND PROOFS PRODUCED WHEN LEARNING TRIGONOMETRY

Jorge Fiallo
Dpto. de Matemáticas
Univ. Industrial de Santander
Santander (Colombia)

Angel Gutiérrez
Dpto. de Didáctica de la Matemática
Univ. de Valencia
Valencia (Spain)

Abstract*. Students usually learn mathematical proof based on contents of Euclidean geometry, calculus or numbers. Trigonometry is usually taught in a routine algorithmic way, but we show that also this topic can be used to teach students to prove conjectures. In this paper we describe a teaching experiment aimed to promote 10th grade students' ability to prove while meaningfully studying trigonometry with the help of a DGS. We present examples of the different types of proofs produced by the students, and show their progression during the teaching experiment.*

INTRODUCTION

The learning of proof is one of the most active research agendas in Mathematics Education. Mariotti (2006) suggests the existence of three research directions into this agenda: Analysis of the roles of proof in mathematics curricula, approaches to students' conceptions of proof, and teaching experiments to teach students to prove. Another very active research agenda is related to the use of new technologies, mainly computers, in teaching and learning mathematics. In particular, dynamic geometry software (DGS) has proved to be an excellent environment to learn geometry.

The integration of both research directions, teaching students to prove with the help of software, shows that computers are a powerful tool that can be successfully used to help students understand the need of mathematical proofs, and to explore, analyze and get data in order to state conjectures and to devise ways to prove them. Computer microworlds provide students with environments where the mathematical concepts are seen as objects that can be handled, transformed and observed.

Most research on teaching mathematical proof are based on Euclidean geometry, and some others on calculus or numbers, but very seldom students are asked to prove trigonometric properties. When studying trigonometry, quite often students just have to memorize a set of identities and to apply them to solve routine exercises. When students are asked to do proofs in trigonometry, proofs usually consist of algebraic transformations linking a side of an identity to the other side. On the contrary, to promote students' understanding of trigonometric concepts, they should be provided with tools and procedures that help them to analyze and relate concepts, to produce and prove conjectures, to meaningfully learn the relationships, concepts or properties.

In this paper we present results from a research aimed to get a better understanding of students' learning of proof processes by observing the ways students prove conjectures in trigonometry. The research was based on the design, experimentation and analysis of a teaching unit aimed to teach trigonometry to 10^{th} grade students in a DGS environment and, at the same time, to induce students to prove the conjectures they get from their explorations.

RESEARCH FRAMEWORK

The analysis of students' answers focused on their abilities to prove the conjectures raised as answer to the activities. We understand *mathematical proof* in a wide sense, including formal proofs but also any attempt made by students to convince themselves, the teacher or other students of the truth of a mathematical statement or conjecture by means of explanations, verifications or justifications.

The analysis was based on the categories of proofs described in Marrades, Gutiérrez (2000) who, elaborating on the types of proofs identified by Bell (1976), Balacheff (1988), and Harel, Sowder (1998), defined an analytic frame-work to characterize students' answers to proof problems. Due to space limitation, we only include here short descriptions of the types of proofs integrating this framework:

A) *Empirical proofs*. The types of empirical proofs are:

* *Naive empirical proofs*, when a conjecture is proved by showing that it is true in examples selected without a specific criterion. Depending on how the examples are used, a naive empirical proof may be:
  - *Perceptual proof*, when it involves only visual or tactile perception of examples.
  - *Inductive proof*, when it also involves the use of mathematical elements or relationships found in the examples.
* *Crucial experiment proofs*, when a conjecture is proved by showing that it is true in a specific, carefully selected, example. A crucial experiment proof may be:
  - *Example-based proof*, when it only shows the existence of an example or the lack of counter-examples.
  - *Constructive proof*, when it focuses on the way of getting the example.
  - *Analytical proof*, when it is based on properties empirically observed in the example or in auxiliary elements.
  - *Intellectual proof*, when it is based on empirical observation of the example, but the justification mainly uses accepted abstract properties or relationships among elements of the example.
* *Generic example proofs*, when the proofs are based on a specific example, seen as a characteristic representative of its class.
  - The four above defined types of crucial experiment proofs are used to discriminate generic example proofs too.

B) *Deductive proofs*. The types of deductive proofs are:

* *Thought experiment proofs*, when a specific example is used to help organize the proof. Depending on how the example is used, a crucial experiment proof may be:
  - *Transformative proof*, when it is based on mental operations producing a transformation of the initial problem into another equivalent one.
  - *Structural proof*, when it consists of a sequence of logical deductions derived from the data of the problem and axioms, definitions or accepted theorems.
* *Formal deduction proofs,* when they do not have the help of specific examples.
  - The two above defined types of thought experiment proofs are used to discriminate formal deduction proofs too.

METHODOLOGY

During the two weeks previous to the beginning of the classes, the first author met each teacher to make them aware of the research aims, teaching objectives and methodology, their expected role as teachers, etc. Our conception of proof in mathematics was specially emphasized to the teachers.

The teachers were the responsible for the teaching, and the researcher acted as a participant observer, taking field notes, observing students' behaviour and collaborating with the teachers by answering some students' questions and queries.

Data gathered during the teaching experiment to analyze students' activity were students' answers to a written diagnostic test to show their previous knowledge and proof abilities, groups' answers written in the activity sheets, concept maps filled in by the groups at the end of several sets of activities, three written exams posed by the teachers during and at the end of the teaching experiment, and videotapes recording daily actions and dialogs of two groups from each school.

THE TEACHING EXPERIMENT

The sample.

The teaching experiment was carried out with 100 grade-10 students (aged 15-16) in three whole classroom mixed ability groups from three secondary schools at Santander (Colombia). The sample was selected on the base of availability of schools and teachers, who showed their interest to collaborate in this experiment. In Colombia, 10^{th} grade is the first year of non-compulsory secondary school, and trigonometry is taught for the first time in this grade. The students from two schools were average students, and those from the third school were above average students. They had never been asked before to prove mathematical statements or conjectures.

The experiment.

The teaching experiment took place as part of the ordinary classes of mathematics, for a period of about 12 week. Each group had two 90 minute classes per week in a computer room using Cabri II+. In two schools the students worked in small groups (2 or 3 students per group) with a computer, and in the other school each student worked with a computer. Students could use Cabri freely to solve all the proof problems posed. They didn't have previous experience in using a DGS. The teaching unit included five activities:

1) Introduction of trigonometric ratios of right triangles.

2) Introduction of trigonometric ratios of angles in standard position.

3) Visualization and vector representations of trigonometric ratios.

4) The Pythagoras Theorem and related trigonometric identities.

5) The sine of the addition of two angles.

Each activity was integrated by a set of related sub-activities. Some examples of sub-activities are shown in next section; space limitation impedes us to include more details about their content.

The experimental teaching unit was designed in a guided discovery teaching style, with the first parts of activities 1 to 3 setting the ground knowledge on trigonometric ratios (both graphical, algebraic, and analytical) to be used in the second parts of those activities and in activities 4 and 5.

As a main objective of the teaching unit was the development of students' proving abilities, they were asked from the very beginning to analyze and prove any conjecture they made. A frequent way for the groups of students to solve an activity was to make a figure in Cabri, or open a file with a figure, to explore the figure looking for a conjecture, to debate this conjecture, to write the group's conclusions and arguments in the activity sheets, and finally to participate in a class discussion with the other groups and the teacher. As the students hadn't used Cabri before these classes, and there was a limited time to teach them to use Cabri, students were only asked to make easy figures, and they were provided with files containing already made figures in the other activities; then Cabri was mainly a tool to visualize and dynamically explore and analyze trigonometric definitions, properties and relationships.

EXAMPLES OF PROOFS OF TRIGONOMETRIC PROPERTIES

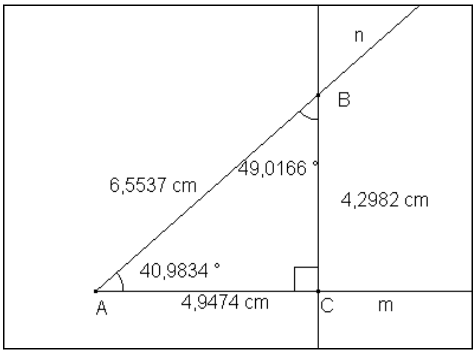
In this section we present examples of different types of proofs produced by the students. It is interesting to note that, even with a rather small sample of students, who had never been asked before to prove conjectures, we obtained a quite large variety of types of proofs, showing that trigonometry may be a rich context to make students engage in learning to prove.

Examples of empirical proofs.

Naive empiricism. In activity 1.3.1 students were asked to create in Cabri a right triangle ABC (Figure 1a), based on two rays *m* and *n* and a straight line perpendicular to *m*, and to get and prove a relationship between the acute angles A and B.

Students in group G1G filled in a table (Figure 1b) with measures for angles A and B taken from the screen and they raise a conjecture: *The measures of angle A and angle B add the same as angle C*. Then this dialog took place:

1 Researcher: Angle C, how much does it measure?
2 Students: 90º.
3 Researcher: Is it the same for any triangle? Is it always true?



- a -

- b -

1	61.9330	28.0670
2	59.2209	30.7791
3	31.4057	58.5943
4	30.4019	59.5981
5	22.2341	67.7659
6	11.4087	78.5913
7	12.3369	77.6631

Figure 1: a) Right triangle for activity 1.3.1. b) Table filled in by students.

4 Students: I think so.
7 Researcher: Then, how would you justify that the addition is really 90º?
8 Students: Because we have several measures here in the table, and if we add them it is ever 90º, any two we take.
9 Researcher: Is this enough to justify it [the conjecture]?
10 Students: Yes.

In this dialog the students showed an inductive naive empirical proof, since the proof is based on the data in the table, collected without any specific criterion.

Crucial experiment. In activity 2.3.1 students were asked to find and prove a relationship between sin(A) and sin(-A).

During the whole class discussion, student C10 explained to the class her answer. After having drawn on the board an acute angle A and angle -A (Figure 2), the student explained:

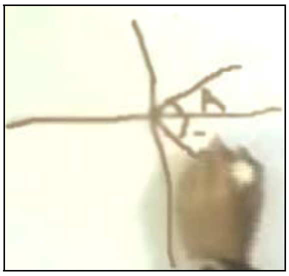


Figure 2: Students C10 drew angles A and -A on the board.

3 C10: Then the values of this [pointing to ∠ A] are the same as the absolute values of this [pointing to ∠ -A].

4 C10: Then here, let's say this point [marking a point on the terminal side of ∠ -A] and this point [marking a point on the terminal side of ∠ A] ...

5 C10: Let's say this [point] is 3 and ... 2 [writing (3,2) next to the terminal side of ∠ A].

6 C10: Then this [point] would be 3 and -2 [writing (3,-2) next to the terminal side of ∠ -A]. Three, minus two [whispering].

7 C10: Then ... we can say that the values, as x is the same because x, as x is ... [pointing to the positive end of axis X] it is shared [by the points on terminal sides of ∠ A and ∠ -A], and y [pointing to values 2 and -2] is ... the absolute values are equal, so sine is supposed to be y over r [writing $\frac{y}{r}$].

16 C10: Then we have that the radius is the same for both [angles, pointing to the terminal sides of ∠ A and ∠ -A].

18 C10: And y is the absolute value is ... the absolute value is equal [pointing to values 2 and -2], but here ... [pointing to the sign of -2].

20 C10: In A it would be $\frac{2}{r}$, and this would be $\frac{-2}{r}$ [writing them on the board].

21 C10: Then, as r is equal, it would be ... I mean the divisions would be equal, only would be different the signs, so the results would be opposite additive or inverse additive.

22 Teacher: Then, what is the relationship between sin(A) and sin(-A)?

23 C10: We have that sin(A) = -sin(-A) [writing the identity on the board].

Student C10's proof was a combination of statements based on particular values of angle A (it was in the 1st quadrant) and coordinates, (2,3) and (-2,3), and attempts to generalize, by using symbols like $\frac{y}{r}$. The arguments she used were correct, but they were based on a specific example. Most likely, when the student was solving the activity with the computer, she dragged the terminal side of angle A and noted that, for any angle A in the circle, angles A and -A had the same abscise and inverse ordinates. Anyway, when explaining her conjecture to the class, she didn't feel the need to mention the angles A in other quadrants. Therefore, this is a case of

<u>intellectual crucial experiment</u> proof.

Generic example. In activity 2.3.2 students were asked to find relationships between the trigonometric ratios for angles A, A-90º and 90º-A, and to probe their conjectures “by using accepted mathematical properties”.

Student F02 was working on a drawing (Figure 3; letters V and W added to make easier the references). Then the following dialog took place (during the dialog the student was pointing to the objects on the screen he was mentioning):

1	Researcher:	What did you find?
2	F02:	I changed a little the figure and added another ... another perpendicular line going through point S, which is ... which is the intersection of the circle and the side of [angle] 90º-A.
3	Researcher:	Ok.
4	F02:	When I did it, ... this ... this triangle, which is A, S, and axis X, appeared [triangle ASW]. I noted that it is equal to triangle A, P and axis X [triangle PAV]. It [line AP] was the first line a drew.

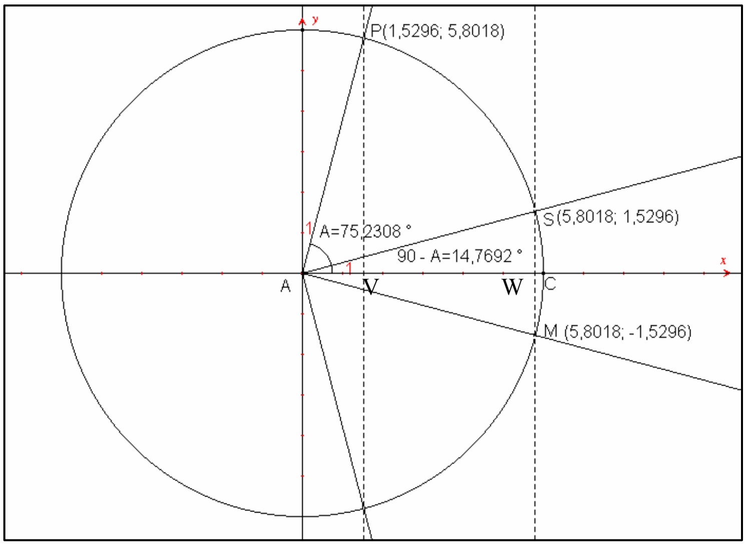


Figure 3: Drawing for activity 2.3.2.

5	Researcher:	Ok.
8	F02:	Then, the angle between S and C [∠ CAS] is equal to the angle between Y and P [∠ YAP], there isn’t much to say about this [i.e., the equality is evident], but I can say that P and C [∠ PAC] is equal to S and Y [∠ SAY].
9	F02:	Why? Because with angle Y the angle between Y and C is 90º.
10	Researcher:	Ok.
11	F02:	Then, when I subtract this angle, angle A, this angle appears [∠ 90º-A].
12	F02:	As the triangles are equal, I can say that the sine of this one [∠ SAW], which is the opposite [side] over the hypotenuse, shall be equal to cosine ... to cosine of A, which is this distance here [AV] over the

hypotenuse. As the hypotenuse is the radius, it will ever be the same and, from the figure I have made, I get that ... these two distances [SC and AV] are equal, so the ... the ... the values of these two ... these two [trigonometric] ratios are equal, equal to sine of A and ... [stopped by the researcher]

13 Researcher: Equal in the signs too?

14 F02: Equal in the sign too [while moving the head up and down].

17 Researcher: Then, we can say that sine of A is equal to ...?

18 F02: To cosine of ... of ... 90-A.

19 F02: And also that cosine of A is equal to sine of 90-A.

Student F02 stated a conjecture starting with an example intended to represent all the angles in the circle; the student didn't check nor mention angles in other quadrants. The proof included several references to the way the example had been made. The student tried to produce abstract arguments, although they really referred to properties or elements of the drawing. The drawing in the screen showed the measures of angles and coordinates of points, but the student didn't use them explicitly. Then F02's proof is a case of <u>constructive generic example</u> proof.

Examples of deductive proofs.

Mental experiment. In a problem included in an exam, the students were asked to find a relationship between cot(360º-α) and cot(α), and to prove it.

Student C16 drew the diagram and then she wrote this proof:

cot(360º-α), cot α

360º - acute = in the 4th quadrant (θ)

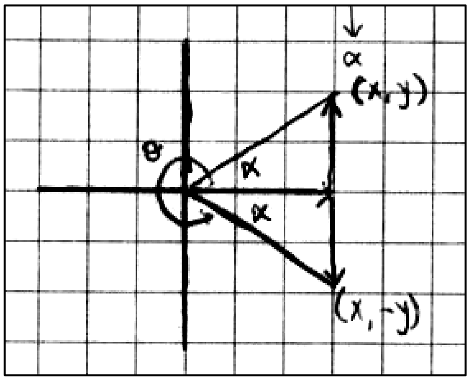
$\cot\theta = \dfrac{x}{-y}$ $\qquad$ $\cot\alpha = \dfrac{x}{y}$

$\cot\theta < 0$ $\qquad$ $\cot\alpha > 0$

$|\cot\theta| = |\cot\alpha|$

$-\cot\theta = \cot\alpha$

$\boxed{-\cot(360^o\text{-}\alpha) = \cot\alpha}$



The student first drew an acute angle α and the angle 360º-α in the Cartesian plane. Then she assigned coordinates (x,y) and (x,-y) to two points in the terminal sides of these angles. She knew that the triangles drawn in the diagram are congruent because this property had been studied in a previous activity. The student also used the label θ to name the angle 360º-α, and then she wrote algebraic transformations to deduce the correct relationship. The diagram drawn only played the role of an auxiliary abstract example, but the student organized the proof of the conjecture based o it. Therefore this is a case of a <u>structural mental experiment</u> proof.

Formal deductive. In activity 3.4.8 students were asked to find a relationship between sin(360º-α) and sin(α), and to prove it. Student C16 wrote this proof:

IV QUADRANT $\sin(360^{o}-\alpha) = \sin\theta$

$$\sin\theta \text{ and } \sin\alpha$$

$$-A \quad \text{and} \quad A$$

$$\boxed{-\sin\theta = \sin\alpha}$$

$$\sin\theta = \frac{y}{r} \qquad \sin\alpha = \frac{y}{r}$$

$$y < 0 \qquad y > 0$$

$$\sin\theta < 0 \qquad \sin\alpha > 0$$

$$-\sin\theta = \sin\alpha$$

First student C16 wrote the conjecture $\sin(360^{o}-\alpha) = \sin\alpha$ and, like in the previous example, she changed $360-\alpha$ to θ. She noted that the terminal side of $\angle 360-\alpha$ is the same of $\angle -\alpha$ and reminded the identity, proved in activity 2, $\sin(A) = -\sin(-A)$. Then she wrote the conjecture $-\sin\theta = \sin\alpha$ and afterwards she proved it by using coordinates of points in the sides of the angles and algebraic expressions. The student assumed that α is an angle of reference in the 1st quadrant, so $360^{o}-\alpha$ (-A) is in the 4th quadrant, and the ordinates associated to these angles have the same absolute value but opposite signs. This is a decontextualized proof based on transforming the initial problem (find a relationship between angles $360^{o}-\alpha$ and α) into another one (find a relationship between angles A and -A), so it is a case of a <u>transformative formal deductive</u> proof.

SUMMARY OF RESULTS AND CONCLUSIONS

Table 1 synthesizes the types of proofs produced by the students in each activity. The cells with thick border represent the most frequent type of proof for each activity.

A ct iv.	F	FE	PNE	INE	ECE	CCE	ACE	ICE	EGE	CGE	AGE	IGE	FD	TTE	STE	TFD	SFD
1																	
2																	
3																	
4																	
5																	

Codes: F = Failed. **FE** = Failed empirical. **PNE** = Perceptive naive empirical. **INE** = Inductive naive empirical. **ECE** = Example-based crucial experiment. **CCE** = Constructive crucial experiment. **ACE** = Analytical crucial experiment. **ICE** = Intellectual crucial experiment. **EGE** = Example-based generic example. **CGE** = Constructive generic example. **AGE** = Analytical generic example. **IGE** = Intellectual generic example. **FD** =

Failed deductive. **TTE** = Transformative thought experiment. **STE** = Structural thought experiment. **TFD** = Transformative formal deduction. **SFD** = Structural formal deduction.

Table 1: Types of proofs produced by the students.

We can note, along the teaching experiment, a change in the types of proofs produced by students, from empirical proofs (in activities 1 and 2) to deductive proofs (in activities 4 and 5). Failed proofs (when students didn't succeed in writing any proof, even a wrong one) were limited to activities 1 and 2. The students making more deductive proofs were those who showed, from the beginning of the experiment, a tendency to present their arguments deductively and also showed a better knowledge of the necessary previous mathematics contents.

The types of proofs produced by the students were also related to the content of the activities. We can observe that most proofs written to answer activity 1 were empirical, partly because students were asked to drag the figures on the screen and to draw conclusions out. The least frequent type of proofs was the generic example, mainly because the kind of activities and questions posed to students didn't promote this type of proofs. Activity 2 was the one with most variety of types of proofs; this may be a consequence of having included in this activity an assessment questionnaire with five proof problems. Activity 3 is the first one having many more deductive proofs than empirical ones; more specifically, the type of proof most frequently produced by the students was the transformative formal deduction; we believe that the help provided to students by the visualization of properties in the Cabri figures was a main reason for this result. In activities 4 and 5, only deductive proofs were produced to solve them, the most frequent types of proofs being the most abstract and formal ones; Anyway, the help of the Cabri figures to visualize conjectures and to suggest the students ways to prove them was decisive in their success. The answers to these activities showed also a clear progress of the students in one of the school towards deductive proofs based on general geometric and algebraic properties induced by the visual proofs suggested by the figures in Cabri.

Finally, without purpose of generalization, we can conclude from this research that trigonometry is a rich field whose teaching can be organized around discovering activities as a way to promote learning with understanding and the development of abilities of proving.

REFERENCES

Balacheff, N.: 1988, 'Aspects of proof in pupils' practice of school mathematics', in D. Pimm (ed.), *Mathematics, teachers and children*, Hodder & Stoughtonpp, London, pp. 216-235.

Bell, A.W.: 1976, 'A study of pupil's proof-explanations in mathematical situations', *Educational Studies in Mathematics* 7(1), 23-40.

Harel, G. and Sowder, L.: 1998, 'Students' proof schemes: Results from exploratory studies', in A.H. Schoenfeld, J. Kaput and E. Dubinsky (eds.), *Research in collegiate mathematics education*, American Mathematical Society, Providence, USA, Vol. III, pp. 234-283.

Mariotti, M.A.: 2006, 'Proof and Proving in Mathematics Education', in A. Gutiérrez and P. Boero (eds.), *Handbook of research on the psychology of mathematics education*, Sense Publishers, Rotterdam, The Netherlands, pp. 173-204.

Marrades, R. and Gutiérrez, A.: 2000, 'Proofs produced by secondary school students learning geometry in a dynamic computer environment', *Educational Studies in Mathematics* 44(1/2), 87-125.

ANALYSIS OF THE TEACHER'S ARGUMENTS USED IN THE DIDACTICAL MANAGEMENT OF A PROBLEM SOLVING SITUATION

PATRICK GIBEL[1]

ABSTRACT: *In this paper, we analyze an investigative situation proposed to a class of 5th graders in a primary school. The situation is based on the following task: In a sale with group rates on a sliding scale, the students must find the lowest possible purchase price for a given number of tickets. The aim of this paper is to show that one of the intrinsic features of the situation restricted the teacher's possibilities of making didactical use of the students' forms of reasoning processes during whole class presentation and discussion of the reports.*

1. INTRODUCTION

The study presented in this paper is a part[2] of an article on the role of the different forms of reasoning in the didactical relation, in mathematics, at the primary school level.

We start by explaining what we mean by "reasoning" (section 2). The term is widely used by teachers of all subjects and by researchers, with a variety of meanings.. Therefore, we had to directly define the object and the methodology of our study before classifying the different forms of reasoning we were concerned with.

In section 3, we will present the problem situation observed and in section 4, we will identify several forms of reasoning which appeared in class during students' investigation [in small groups] and subsequent whole class presentations and discussions.

In section 5, we will address the following questions:

Did the proposed problem situation favor students' production of forms of reasoning? Which didactical decisions of the teacher strongly determine the presence, the meaning and the actual possibilities of processing and using students' forms of reasoning?

2. REASONING IN THE CLASSROOM

2.1. Actual forms of reasoning

We define a *reasoning* as a relation *R* between two elements A and B such that,

- A denotes a condition or an observed fact, which could be contingent upon particular circumstances;

---

[1] Laboratoire DAESL, Didactique et Anthropologie des Enseignements Scientifiques et Langagiers, Université Victor Ségalen-Bordeaux 2 et IUFM d'Aquitaine, France.

[2] It is based on a set of conceptions and results which have been presented in more detail in Guy Brousseau and Patrick Gibel:"Didactical handling of students' reasoning processes in problem solving situations", Educationnal Studies in Mathematics (2005) 59; 13-58.

- B is a consequence, a decision or a predicted fact;
- *R* is a relation, a rule, or, generally, something considered as known and accepted. The relation *R* leads the acting subject (the reasoning "agent"), in the case of condition A being satisfied or fact A taking place, to make the decision B, to predict B or to state that B is true.

An *actual* reasoning contains, moreover,
- an agent E (student or teacher) who uses the relation *R*;
- a project, determined by a situation S, which requires the use of this relation.

We can say that to carry out a project determined by a situation S the subject uses the relation *R* which allows him to infer B from A. This project can be acknowledged and made explicit by the agent, or it can be attributed to him by the observer on the basis of some evidence.

2.2 First classification of forms of reasoning according to their function and type of situation

As implied in the previous section, reasoning is characterized by the role it plays in a situation, i.e. by its function in this situation. This function may be to decide about something, to inform, to convince, or to explain. The function of reasoning varies according to the type of situation in which it takes place; on whether it is a situation of action, formulation, validation or other (Brousseau, 1997: 8-18).

3. THE OBSERVED LESSON

3.1 The components of the situation

The lesson took place in a 5th grade mathematics class.

3.1.1 The problem and the objective situation

The teacher starts by handing out the following problem:

> A one-day ski trip to the resort of Gourette is being organized next Saturday for students from the Oloron area. For this exceptional event, the local city council has decided to pay for the ski passes for the day. The resort of Gourette offers the following group rates:
> 216 passes: 1275F
> 36 passes: 325F
> 6 passes: 85F
> 979 children have signed up for the trip but when the morning of departure arrives 12 children do not turn up because they are sick, of course. The council accountant says to himself "Too bad for these kids, but never mind, it'll work out less expensive for us this way".
> What do you think?

The "objective situation" is the situation presented in the problem; the student is expected to deal with it without questioning the status of reality or not of what is thus presented to him as "objective".

3.1.2 The planned phases of the lesson

The development of the lesson, chosen by the teacher, follows a plan that has become quite common in France:
- the research activity is presented by the teacher (phase 1);
- students read the problem (phase 2);
- the teacher provides additional information, if necessary; for example - explains the terms used in the formulation of the problem (phase 3),
- students work on the problem individually for about 10 minutes (phase 4),
- students are divided into small groups (phase 5),
- students work in small groups, and prepare a written report; this phase (phase 6), lasts about 25 minutes;
- whole class presentation and discussion of the reports, with each group going to the board in turn to present their results (phase 7).

3.2 How the lesson developed

3.2.1 The research activity and the written traces of it In the observed lesson, the research activity was based on the research and formulation of the question, which completely determines the problem (in the classical sense of the term). But the students were not able to perceive what is at stake (mathematically) in the problem situation and it is the teacher himself who formulated the question: "When, do you think, is the ski trip more expensive: when there are 979 students or when there are 967 students?"

3.2.2 The phase of whole class presentation and comparison of students' solutions

Our theoretical, *a priori*, analysis of the problem situation led us to expect a failure of the teacher's plan: The management of the didactical phase of the lesson (phase 7) appeared all the more delicate that the reduction of the complexity was essentially in the hands of the teacher; it depended on his choices, his decisions and his "opportune" interventions.

But upon viewing the video recording of the lesson (which we haven't seen before the theoretical analysis), we had to admit that the teacher managed to conduct his class without being challenged with any major difficulties.

4. THE OBSERVED FORMS OF REASONING, THEIR FUNCTION AND USE

4.1 Forms of reasoning in students' written productions

The analysis of the different forms of reasoning which appear in the students' solutions shows that what is really at stake in the problem situation, namely the problem of minimizing the expense, has not been grasped by the majority of students.

In this lesson, it is clear that the devolution of the situation did not work; the students were not able to take charge of the proposed situation. Indeed, in the phase of whole class discussion and comparison of solutions, it appears that:

- The students do not possess the necessary knowledge to conceive of the basic strategies.
- The students cannot obtain, as feedback to their actions, the information necessary for the solution of the problem.
- There is not enough time for the students to produce a solution, because of the complexity of the problem.
- The students have no means to judge, by themselves, the validity of their solutions.

4.2 Analysis of an episode of interactions during the whole class discussion phase

For this paper, we have chosen to present an analysis, in terms of the theory of didactical situations in mathematics, of an excerpt from the transcript of phase 7, i.e. the whole class discussion and comparison of students' solutions phase.

The episode focuses on interactions related to one student's work. This student, Julien has chosen to work alone. His written work is presented in Figure 1.

Our analysis of this episode is presented in Table 1. The first column of the table contains the code of the intervention, where the first number (4) indicates that Julien's "small group" (composed of him alone) was the fourth to present its results. For some interventions, the timing is shown (since the beginning of phase 7). The second column contains the transcript, and the third some comments on the intervention. In the fourth column we analyze the nature and the function of the intervention with regard to the locutor's intended project. The fifth column aims at articulating the function of the intervention.

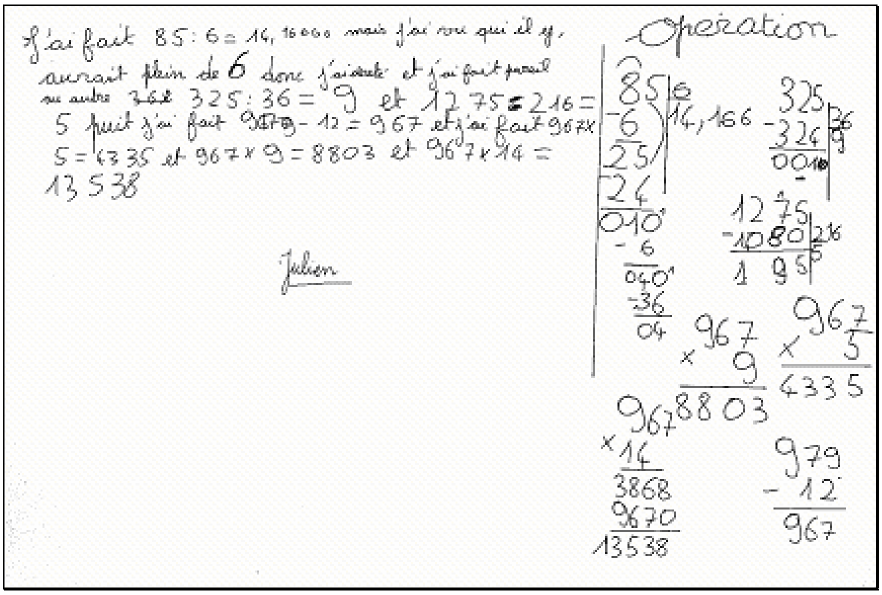


Figure 1

TABLE 1. Transcript and analysis of some interactions

N° Min.	**Transcript**	**Comments**	**Analysis**	**Nature and function of the intervention**
4.1 12'35	Julien: Okay, I started by doing… (1) I divided 6 into 85 (2) and I got 14,166;	Julien comes to present his work. He describes his calculation, without defining or naming the		(1)Direct description of an action (calculation) (2)Formulation of a result (3) Indirect reference to an action: by analogy (4) Organization of the calculation

	I took the 14 and then I saw… (3) I did the same with 325, in short, I did the same, (4) I did the same with all three operations	variable that he calculates.		Strategic or organizational reasoning, local and expressed orally.
4.2	Teacher: (1) the three proposals, (2) the group passes	The teacher reformulates a part of the student's statement to introduce a vocabulary.	The teacher wants to establish a link between the performed calculations and the objective situation.	(1) Correction of the terminology (2) Suggestion of a terminology and giving a name to a result.
4.3	Julien: (1) 325 divided by 36 and 1275 divided by 216 (2) and then I did…	Julien continues to describe his calculations		(1) Direct description of an action (2) Organization of calculation. Strategic or organizational reasoning, local and expressed orally.
4.4	Teacher: (1) [Your] first conclusion after these calculations? [to the whole class] Have you heard the operations he had done? (2) What is the price of a pass, relative to each of the three proposed conditions, right?	The teacher asks Julien what he got from the calculations he performed. He intervenes to provide an interpretation of the calculations. He points to the nature of the results as the "price of a pass relative to each of the three conditions".	The teacher gives an interpretation of each of the calculations performed by Julien. His didactical intention is to construct Julien's calculations as a support for introducing the stages of reasoning .	(1) Giving a statement the status of a "conclusion" in the development of a reasoning. Invitation to comment on Julien's results and to position them relative to an action. (2) Use of rhetorical didactical means:
4.5	Julien: Yeah!			Agreement, approval.
4.6	Teacher: (1) Okay, first conclusion after that?	The teacher questions Julien on what he gets from his calculations.	.	(1) Request to make an inference. The teacher waits for the student to continue his reasoning and articulate a conclusion.
4.7	Julien: And then I did…	No answer; Julien seems to want to continue to describe his calculations.		
4.8	Teacher: No, your first conclusion after that? When you were done with these calculation, what did you think to yourself?	The teacher reiterates his question.	The teacher makes a second attempt, with the same aim as in 4.4. But the formulation is more precise.	Recall of what is a conclusion; invitation to comment on a result.

4.9	Another student: Which one was less expensive.	A student puts into words the question that the teacher has previously asked in an implicit manner.	A student points to Julien what he could get from his calculations, namely a comparison of prices.	Question on an order relation. Project formulation.
4.10	Julien: Yeah! Which one was less expensive… But, no, I couldn't see…			But "which one" does not denote a well-determined object. A passive explanation. Impossibility to realize a project.
4.11	A student: But yes, you can see!	A student points out to Julien that he has all the necessary information.	The student pushes Julien to produce a reasoning, by pointing out to him that he has all the necessary elements to conclude (i.e. to compare the prices).	Possibility of realizing a project.
4.12	Julien: (1) Yes, it was 1275 (2) because a pass cost 5F (3) more or less and then (4) so then I tried, in short, I did 979 less 12, I got 967 and then I multiplied 967 by all the results of the divisions.	Julien gives the expected answer and continues to describe his calculations.	Julien articulates the conclusion, expected in the module 2. He goes back immediately to his initial reasoning, in describing his calculations.	(1) Implicit conclusion (2) Explanation (3) Estimation (4) Direct description of a sequence of actions and organization. Strategic or organizational reasoning, lexpressed orally.
4.13	Teacher: To find what?	The teacher questions Julien on the aim of his calculations.		Project; request to name a result. Request for an explanation.
4.14	Julien: To find the price of how much it was going to cost.	Julien points to the aim of his calculation: to calculate the total expense (for the students who were present at the trip).	Julien indicates the purpose or project he has in mind: for each group rate, to calculate the total expense.	Naming the result. Articulation of the purpose of his procedure.
4.15	Teacher: Yes, the price… to find which one was the least expensive.		The teacher starts from the formulation of the student and transforms it. Julien stated that his aim is to calculate the total expense for each of the three cases. But the teacher focuses on the comparison of the group rates. . The teacher will establish that Julien's	Rhetorical didactical means: Element of a local explicit reasoning of the teacher, which aimes at re-positioning the calculations in the perspective of the comparison of the three rates. Recall of the necessity to subordinate a result to the main task. Didactical intention: reject the calculations by making them appear as useless, redundant, with respect to the previously established conclusion.

			calculations are useless for the comparison of the rates.	
4.16	Julien: Yes.			Accord
4.17	Teacher: And you did the three calculations?	The teacher wishes to make Julien aware of the fact that the calculations were not necessary, that reasoning could help to avoid doing calculations.		Effectiveness of an action.
4.18	Julien: Yes.			
4.19	Teacher: It was necessary?			Call for a judgement of the relevance or adequacy of a calculation.
4.20	Julien: Well... yeah...			Agreement
4.21	Another student: To see which one was the least expensive.			Subordination recall, as in 4.15.
4.22	Teacher: You didn't know it before?	The teacher wants to incite Julien to reflect on his reasons for doing the calculations.	The intention would be: "could you know it beforehand, without doing the calculations?" It is, therefore, a call for a direct reasoning.	Call for the anticipation of the role of a result in the resolution of a problem. Call for a formulation of a direct local reasoning.
4.23	Julien: Yes, I knew it… but…	The student cannot distinguish between his opinion and the justification required by the teacher.		
4.24	Teacher: Okay then, so what is the result?	The teacher re-asks Julien to formulate his conclusion.		
4.25	Julien: So I saw which one was the least expensive, and then…			Validity status: subjective certitude

4.3 Discussion

The analysis of the implicit model of action allows us to identify the implicit mathematical model and Julien's representation of the objective situation. His model is that of the classical commercial situation, based on selling the passes per unit, corresponding to the mathematical model of proportionality.

The transcript (Table 1) shows that, in phase 7, Julien describes his calculations without providing the class with more explanations on why he did them. This is why his project is not accessible to the class, which makes it necessary for the teacher to intervene. By proceeding this way, he presents the teacher with the opportunity to interpret his calculations in a way which does not necessarily correspond to his (Julien's) initial project. The teacher grasps at this opportunity; using rhetorical didactical means, he manages to divert Julien's initial project to the benefit of his own, which is to develop the reasoning underlying Module 2 (comparison of the three rates) of the standard solution.

Moreover, our analysis shows that the teacher tries, several times, to engage a discussion on the validity of the presented procedures, or, more precisely, on the validity of the decisions underlying students' reasoning. However, his attempts all fail, one after another.

5. CONCLUSIONS AND CONJECTURES

5.1 Students' reasoning

The object of our analysis was the influence of certain features of the situation proposed to the students on the elaboration of the different forms of reasoning, their use and the possibilities of their processing available to the teacher during the whole class presentation and discussion of the solutions phase.

This analysis (see Table 1) shows that the forms of reasoning elaborated by the students were few, that they were not very complex in terms of the number of calculations and the number of stages involved.

This analysis implies that the teacher has no means for an effective processing of the produced reasoning, i.e. he cannot use logical reasoning directly related to the objective situation in arguing with the students' solutions.

This brings us to the first conjecture: the factor which constraints the teacher's possibilities of taking into account, articulating and processing students' reasoning is not so much the complexity of this reasoning but another feature which is related to the very nature of the situation proposed to the students.

5.2 The effect of the lesson on students' behavior and learning

5.2.1 The effect of the lesson on the validity of the reasoning and students' conviction

In the complete analysis of the transcript there is a lot of evidence that the students, having produced a reasoning based on a representation conforming to the teacher's expectations, have not become aware of the conditions which define the objective milieu. Indeed, in phase 7, they are unable to formulate the reasons that led them to elaborate these forms of reasoning, or even to react to the reasoning of their classmates when these are based on erroneous representations of the objective situation.

This can be partly explained by the fact that the situation does not provide the students with the possibility of testing their decisions: the objective milieu does not respond with any feedback to the students' actions. Therefore the students have no means to validate or reject their reasoning and therefore to reflect on the decisions underlying their implicit models of action or their representations of the objective milieu.

5.2.2 The effect on the actions, language and opinions of the students

The students, unable to judge the validity of their work, cannot use the reasoning they have produced as arguments in a debate. The debate amongst peers wished for by the teacher is out of the students' reach.

5.3 The effect on the didactical process

5.3.1 The devolution

Decisions underlying the elaboration of each of the models are closely linked with the students' representations of the objective milieu. But this situation is not happening in real time and the students have to imagine the rules governing its functioning. Since the objective milieu is not clearly defined, this leads the students to construct different representations of the situation and therefore also different implicit models of action. Thus, the objective situation cannot be devolved to the students, i.e. the students cannot challenge the retail sales model adopted by the majority, or even calculate the results of the different possible choices.

5.3.2 Didactical corrections

The complete analysis of the transcript shows that the teacher cannot bring the students to articulate the reasons underlying their implicit models of action. To avoid a block, related to the fact that the students do not understand the decisions made by their peers, the teacher is forced to use rhetorical didactical means (Table 1). These means make it possible for the teacher to divert the initial project of a student to the benefit of his own, i.e. the establishment of certain modules of the standard solution. However, the real reasons that justify the elaboration of the module are not there for the students to see; the reasons which underlie and justify the connections between the data given in formulation of the problem situation are hidden.

6. FINAL CONCLUSIONS

The study shows that although the students, faced with a problem situation elaborated and conducted by the teacher, have certainly produced forms of reasoning, they have not made much progress in their practice of reasoning. Indeed, they have not reflected back on their reasoning, on its validity, relevance or adequacy because the teacher was not able to process it. He could not respond to this reasoning by logical arguments based on the objective situation; he was forced to use rhetorical means.

Now, it is not the complexity of the students' reasoning that forced the teacher to use this type of means but the fact that the problem situation could not be devolved to the students. This implies that it is not the teacher's management of the whole class presentation and discussion of the students' work that is challenged here, but rather the nature itself of the situation set up by the teacher, which strongly constrains the possibilities of really taking into account the students' reasoning.

The objective situation does not make it possible for the teacher to bring the students to:

- share with their peers the real reasons that have led each of them to construct implicit models of action and take some decisions in the framework of the corresponding models;
- grasp the reasons why the steps of the expected, standard solution are necessary;
- share the reasoning underlying each module of the standard solution.

If a situation provides the teacher with the possibility of devolving to the students an "autonomous" (or "self-contained") situation of action, then, according to the theory of didactical situations in mathematics, during the phase of analysis of students' solutions the teacher can refer to the objective situation. This is because the students can develop their personal strategies and forms of reasoning related to the situations with which they are confronted. The teacher does not have to have recourse to rhetorical didactical means to process students' forms of reasoning.

If, on the other hand, the teacher has no such possibility, the teacher cannot refer in his arguments just to the objective situation and must bring in information and provide feedback on the basis of a project that is not visible for the students; and this is why he is forced to use rhetorical didactical means.

REFERENCES

BALACHEFF, N., Processus de preuve en situation de validation , *Educational Studies in Mathematics* 18/2, 147-176.

BROUSSEAU G., (1986), Fondements et méthodes de la Didactique des Mathématiques*,* *Recherches en Didactique des Mathématiques*, Volume 7(2), Edition La Pensée Sauvage, Grenoble, pp 33-115.

BROUSSEAU G., GIBEL P.:1999, Analyse didactique d'une séquence de classe destinée à développer certaines pratiques du raisonnement des élèves*,Actes de la X° Ecole d'été de Didactique des Mathématiques,* (2), 54-71.

BROUSSEAU G., GIBEL P (2005) "Didactical handling of students' reasoning processes in problem solving situations", Educationnal Studies in Mathematics 59; 13-58.

BROUSSEAU G., (1997), *Theory of Didactical Situations in Mathematics*, Kluwer Academic Publishers.

OLERON P., (1977), *Le raisonnement*, Presses Universitaires de France.

PERELMAN C., (1970), *Le champ de l'argumentation*, Presses Universitaires de France.

ROBRIEUX J.J. (1993), *Eléments de Rhétorique et d'Argumentation*, Dunod

STRUCTURAL RELATIONSHIPS BETWEEN ARGUMENTATION AND PROOF IN SOLVING OPEN PROBLEMS IN ALGEBRA

Bettina Pedemonte

Istituto per le Tecnologie Didattiche – CNR Genova

This paper concerns a work-in-progress study analysing cognitive continuities and/or distances between argumentation supporting a conjecture and its proof in solving open problems in algebra. There is usually a cognitive distance between these argumentations and algebraic proofs, not only in the structure (algebraic proofs are often characterised by a strong deductive structure) but also in the "content". The aim of this paper is to show this cognitive distance and the role of abductive argumentation to decrease this distance. Toulmin's model is used as a tool to analyse and compare the structures of argumentation and proof.

INTRODUCTION

This paper analyses cognitive continuities and/or distances between argumentation supporting a conjecture and its proof in solving open problems in algebra. This study, developed as part of the ReMath project (IST - 4 - 26751), can be considered as an extension of a previous research work, studying the relationships between argumentation supporting a conjecture and its proof in solving open problems in geometry (Pedemonte, 2002).

Argumentation supporting a conjecture, developed during the resolution process of an open geometrical problem is often characterized by abductive structure which sometimes remains present in the subsequent proof (Pedemonte 2002). Some experiments highlighted that this structural continuity between abductive argumentation and "abductive proof" does not help students to construct a deductive proof. On the contrary, this "natural" continuity can be considered one of the possible troubles met by students in the construction of a proof.

My research interest is in studying the possibility to extend these research results to other mathematical domains. In particular, in this paper, I consider the resolution processes of an open problem in algebra asking for producing a conjecture and constructing of a proof. The aim of this analysis is to see if there is a "natural" structural continuity between argumentation and proof, which can be considered as one of the possible difficulties met by students in the construction of an algebraic proof.

To perform this analysis I put forward a case study. The experiment has been carried out with students of Formation Science University in Genoa. In this paper, two students resolution processes are presented; their argumentations and proofs are analysed by means of Toulmin's model.

COGNITIVE CONTINUITY AND/OR DISTANCE BETWEEN ARGUMENTATION SUPPORTING A CONJECTURE AND PROOF

Some research studies about argumentation and proof highlight the continuity that exists between argumentation as a process of statement production and the construction of its proof; what is in play is the relationship between conjecturing and looking for a proof (Boero, Garuti, Mariotti, 1996). This continuity is called *cognitive unity*. During a problem solving process, an argumentation activity is usually developed in order to produce a conjecture. The hypothesis of *cognitive unity* is that in some cases this argumentation can be used by the student in the construction of proof by organising in a logical chain some of the previously produced arguments.

Experimental research about cognitive unity (Boero & al., 1996; Garuti & al. 1996; Garuti & al. 1998; Mariotti, 2001) shows that proof is more “accessible” to students if an argumentation activity is developed for the construction of a conjecture. The teaching of proof, which is mainly based on “reproductive” learning (proofs are merely presented to students, they do not have to construct them) appears to be unsuccessful. A didactical consequence of this study is that suitable open problems (Arsac, Germain & Mante, 1991) which call for a conjecture could be used to introduce the learning of proof.

Contributing to this research, a theoretical framework has been developed (Pedemonte, 2002) to analyse and to compare argumentation supporting a conjecture and its proof in solving open problems in geometry. This comparison may be carried out by analysing the continuity or the distance between this argumentation and its proof under two points of view: the *referential system* (Pedemonte, 2005) and the *structure* (Pedemonte, 2007). By *referential system* I mean both the representations system (the language, the heuristic, the drawing) and the knowledge system (conceptions, theorems) of argumentation and proof. By *structure* I mean the logical cognitive connection between statements (abduction, induction, or deduction). For example, there is continuity between argumentation and proof in the referential system if some words, drawings, theorems used in the proof have been used in the argumentation supporting the conjecture. There is a structural continuity between argumentation and proof if some abductive steps used in the argumentation are present also in the proof. Otherwise, if argumentation structure is abduction and proof is deduction there is a structural distance between the two.

Research results carried out by this study (Pedemonte, 2007) highlight the importance of structural analysis between argumentation and proof. This analysis shows that although there are cases of continuity in the referential systems between argumentation supporting a conjecture and its proof, it is often necessary to cover a structural distance between the two (from an abductive argumentation to a deductive proof). This structural distance is not always covered by students, who sometime produce incorrect proofs because they are not able to transform the structure of argumentation in deductive structure for proof (Pedemonte 2007).

These research results are limited to the geometrical domain, which is the mathematical domain where usually learning of proof is introduced. Nevertheless, it could be interesting to analyse if it is possible to extend such results to other mathematical domains. In this paper, algebra is considered.

THE ROLE OF ALGEBRA IN PROVING PROCESS

The solution of open problems in algebra asking for a conjecture seems to be usually characterized by two particular phases: the *constructive argumentation* (Pedemonte, 2002) phase that corresponds to the construction of a conjecture (sometime only characterized by numerical examples); the proof phase that concerns the systemic application of algebraic rules, in which each step of the proof is the transformation of the previous step according to a given rule. During the resolution process it is possible to produce another type of argumentation, the *structurant argumentation* (Pedemonte, 2002), which is constructed to justify a conjecture, in particular when the conjecture is constructed as a "fact". I think that this argumentation can play an important role in the resolution process of open problems in algebra.

As a matter of fact, some cognitive research about the resolution of algebraic problems (Duval, 2002) highlights the cognitive gap between the *conversion phase* (or the constructive argumentation phase), i.e. the translation of the problem in algebraic characters, and the *treatment phase* (or the proof phase), i.e. the deduction of the unknown value of the algebraic expression. According to Duval, this gap has to be coped with by students in solving problems in algebra.

Moreover, if we consider that sometime argumentation in open problems in algebra is characterized by explorations based on arithmetic numerical examples, the gap between constructive argumentation and proof is also present as methodological aspect (Chevallard, 1989). Arithmetic moves from known to unknown while algebra often moves from unknown to known in way that at the end of the process it is always possible to know the unknown quantity. Arithmetic and algebra have two separate languages: the first one is based on the ordinary language enriched by a numerical language while the second one is essentially oriented to computation where there is a mechanic control. Other research studies highlight difficulties in catching the invariance of algebraic denotation respect to the sense (Arzarello & al., 1994, Drohuard, 1992); in arithmetic this invariance is automatic because denotation is a specific number while in algebra it is connected to the syntactic aspects.

Following this research results, I make the hypothesis that *structurant argumentation* could be useful to decrease the cognitive gap between the *constructive argumentation* and the proof. In particular, a successful structurant argumentation should favour the continuity in the referential system between constructive argumentation and proof.

From a structural point of view my hypothesis is that in solving open problem in algebra, the "natural" structural continuity between argumentation and proof is usually not present. The connection between two steps in algebra is characterized by a “strong” deductive structure: algebraic expression as equations are modified according to computation rules often implicit for student. This is not always the case for argumentation supporting a conjecture. Argumentation structure can be abductive, or inductive if conjecture is constructed as generalisation on numerical examples.

The problem is that the structural distance between argumentation and proof contributes to increase the gap between the two and sometime student is not able to reconstruct reasoning used to construct the conjecture.

This hypothesis will be illustrated in the next section, where I present the analysis of two student protocols. To complete this discussion, I am going to introduce Toulmin’s model as a tool to analyse proving process performed by students.

Toulmin’s model: a methodological tool to analyse argumentation and proof

As methodological tool to analyse and to compare argumentation and proof I use Toulmin’s model (Toulmin, 1958). In this model argumentation, as proof, has a ternary structure. This fact allows us to compare the structure of argumentation with the structure of the proof.

In Toulmin’s model an argument comprises three elements (Toulmin, 1958/993):

C (claim): the statement of the speaker,

D (data): data justifying the claim C,

W (warrant): the inference rule, which allows data to be connected to the claim.

In any argument the first step is expressed by a standpoint (an assertion, an opinion). In Toulmin's terminology the standpoint is called the claim. The second step consists of the production of data supporting the claim. The warrant provides the justification for using the data conceived as a support for the data-claim relationships. The warrant, which can be expressed by a principle, or a rule, acts as a bridge between the data and the claim.

The basic structure of an argument is presented in Figure 1.

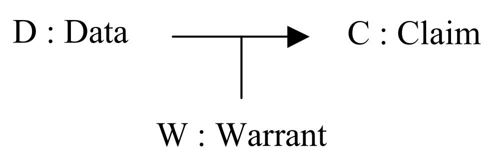


Figure 1: Toulmin’s basic model

Three auxiliary elements may be necessary to describe an argument: a qualifier, a rebuttal, a backing (Toulmin, 1958/993). These elements are not significant for the analysis treated in this paper and for this reason will not be presented.

In Toulmin's model a step appears as a deductive step: data and warrants lead to the claim. Nevertheless, it could be useful to represent other argumentative structures using this model. In this paper we consider abductive structure.

Abduction has been introduced by Peirce (Peirce, 1960) as a model of inference used in the discovery process. According to Peirce, starting from an observed fact, a rule can be supposed, through which the hypothesis becomes more credible. The hypothesis is the conclusion of a reasoning giving it a plausibility value (Peirce 1960, 2.511n). So abduction is a plausible reasoning (Polya, 1962) which can be modelled as follows (Polya 1962, p. 107):

If A then B
B true
___
A more credible

By this scheme we can represent an abductive step in Toulmin's model as follows:

$$\text{D : ?} \longrightarrow \text{C : B}$$

$$\text{W : A} \Rightarrow \text{B}$$

Figure 2: Abductive argumentation in Toulmin's model

The question mark means that data are to be sought in order to apply the inference rule justifying the claim.

Drawing on Toulmin's model (Toulmin, 1958), I analyse structural continuities and structural distances between argumentation and proof.

CASE STUDY

In this section two resolution processes of an open problem in algebra are presented. They are taken from a set of data collected with prospective primary school teachers attending a math course at the University. Students were asked to solve the problem aloud, they worked alone under the supervision of a researcher who do not intervene in helping them. The observation was conducted out of the usual schedule. Students' mathematical background was not homogeneous because students came from different schools. Nevertheless, all of them could solve the problem with their theoretical algebraic background, even if they were not familiar with problems of this kind.

The problem presented to students is the following:

"What can you say about $(p-1)(q^2 -1)/8$ if p and q are odd numbers?"

This is a classical problem, analysed by different research studies (Arzarello & al. 1994, Garuti & al. 1998).

I transcribe the main part of two resolution processes which are based on the transcriptions of the audio recordings and the written productions of the students.

Two examples are presented:

- Example 1: Example of structurant argumentation which decreases the gap between constructive argumentation and proof

- Example 2: Example of structurant argumentation which does not decrease the gap between constructive argumentation and proof

In order to analyse the argumentation, I have selected the assertions produced by students and reconstructed the structure of the argumentative step: claim C, data D and warrant W. The indices identify each argumentative step. The student's text is in the left column, and my comments and analyses are reported in the right column. The texts have been translated from Italian into English.

Example 1

Manuela constructs the conjecture as generalization of numerical examples. The structure of this argumentation is inductive and the referential system is based on arithmetic.

If p=11 and q=13 then …*(She calculates)* the result is 210 If p=7 and q=9 then the result is … 60 They are even numbers Then probably $(p-1)(q^2-1)/8$ is an even number	**probably** **D_1: Previous arguments** → **C_1: The expression $(p-1)(q^2-1)/8$ is an even number** **W :generalisation** *Conjecture is a fact constructed as generalization on numerical examples; now student has to justify it.*

Manuela produces a structurant argumentation to justify her conjecture. She analyses expression $(p-1)(q^2-1)/8$ considering even and odd numbers properties. She is not able to conclude.

if p is an odd number, p-1 is even; if q is an odd number, q^2-1 is an even number too, Then an even number times an even number is an even number, then the expression is an even number…	**D_2: p is an odd number** → **C_2 : p-1 is an even number** **W : even and odd number property** **D_3: q is an odd number** → **C_3 : q2 -1 is an even number** **W : calculus rule and even and odd number property** **D_4: C_2, C_3** → **C_4 :$(p-1)(q^2-1)$ is an even number** **W : the product of two even numbers is an even number**

Manuela understands that the claim C_4 is not sufficient to justify the conjecture. She looks for another element allowing her to state the conjecture.

… but there is the division by 8. I have to find something else to say that the expression is even.	**D_5: C_4 and ? → $C_5 = C_1$: (p-1)(q^2-1)/8 is an even number** **W : calculus rule and even and odd number property** *This is an abductive structure.*

The last argument is very important in the structurant argumentation because it leads Manuela to look for something else (the question mark in D_5) to justify the conjecture. She analyses q^2-1. By some numerical examples Manuela understands that q^2-1 cannot be less than 8. She does not say explicitly that q^2-1 is divisible by 8 but we can suppose she makes this consideration because she considers different values for q (1, 3, 5, 7, 9)

But q^2-1 cannot be equal to 2 neither 4 because with 1 q^2-1 is 0 and with 3 q^2-1 is 8. Then the minimal number is 8; 8 over 8 is 1 then the expression is an even number.	**D_6: Substitution in q2 -1 of some numerical examples (q=1, q=3) → C_6: q^2-1 cannot be equal to 2 or to 4** **W : calculus** **D_7: C_6 → C_7 : q^2-1 cannot be less then 8** **W : comparison among numerical substitution in the formula** *Manuela can complete data 5 by the claim C_7.*

We observe that the structurant argumentation is characterised by both arithmetic and algebraic reasoning. Manuela looks for elements useful to construct the proof.

q = 2n+1 then q^2-1 = $4n^2$+4n+1-1 = 4n(n+1). This is at least divisible by 4, and so what remains is n(n+1), which is surely divisible by two, because if n is even everything is fine, if n is odd (n+1) is even and then 4n(n+1) is at least divisible by 8. We may conclude that q^2-1 is a multiple of 8. Then (p-1)(q^2-1)/8 is even if p and q are odd.	**D_8: q = 2n+1 → C_8: q^2-1 = 4n(n+1)** **W : substitution** *Manuela analyses the formula 4n(n+1) to prove that q^2-1 is at least 8.* **D_9: n is an even number → C_9: 4n(n+1) is at least divisible by 8** **W : calculus rule (4*2t=8t with 2t=n)** **number … least divisible by 8** **W : n+1 is an even number and calculus rule**

Arguments 9 and 10 allow concluding that 4n(n+1) is always at least divisible by 8. Then the argument 5 is transformed into a deduction step and the conclusion is carried out rapidly. We can observe that referential system is based on arithmetic for the constructive argumentation and on algebra for proof. The structurant argumentation contains arithmetic and algebraic elements allowing the continuity in the referential system between the two. It is the abductive step in structurant argumentation which allows the connection between the constructive argumentation and the deductive proof: Manuela analyses the expression 4n(n+1) to prove that this expression is divisible by 8.

Example 2

Let's consider the answer produced by another student, Elio. He tries different strategies: at the beginning he produces a reasoning similar to the previous one (the arguments 2, 3 and 4 of the previous example) concluding that the expression $(p-1)(q^2-1)$ is an even number. Nevertheless, he says that this fact it is not useful "because in general it is not true that an even number divided by another even number makes an even number". Then he assigns some numbers to the letter p and q and by means of a generalisation he constructs conjecture.

If p=1 and q=3 then 0*8/8=0 p=5 and q=7 then 4*48/8=24 p=11 and q=13 then 10*168/8=210 It seems that the expression (p-1)(q2-1)/8 is even.	**probably** **D_1: Previous arguments** → **C_1 : The expression $(p-1)(q^2-1)/8$ is an even number** **W :generalisation** *Conjecture is constructed as generalization based on numerical examples.*

As in the previous example conjecture is based on arithmetic examples. These examples allow concluding that q^2-1 seems to be divisible by 8.

And…. Wait… It seems that by substituting q with an odd number, q^2-1 is divisible by 8. Then $(p-1)(q^2-1)/8$ is even because p-1 is even and q^2-1 is divisible by 8	**D_2: Claims based on numerical examples** → **C_2: q^2-1 is always divisible by 8** **W : generalisation** *Elio concludes that q^2-1 is divisible by 8. Then he justifies the conjecture:* **D_3: p-1 is an even number and C_2** → **C_3: C_1** **W : calculus rules and even and odd number properties**

Elio has constructed a structurant argumentation which allows him to justify conjecture. Nevertheless this justification is still based on arithmetical examples. Moreover, there is no abductive step to connect constructive argumentation with proof. As a matter of fact, Elio tries to construct a proof but without any result. He looses the connection with the argumentation phase; he is driven by deductive structure of algebraic proof.

I try to prove the statement p and q are odd, then p =2k+1 and q=2h+1 then I can find (2k+1-1) $[(2h+1)^2 -1]/8$ $2k[(4h^2+1+4h)-1]/8$ $2k(4h^2+4h)/8$ which is not equal to 2 times something… I cannot conclude I can simplify $k(4h^2+4h)/4$ If I factor 2...$2k(2h^2+2h)$ /4 no... If I factor h: 2kh(4h+4) /8 no… If I factor 4h: 2k*4h(h+1)/8 no I cannot prove with algebra, but I'm sure that the expression is an even number	**D_4: $(p-1)(q^2-1)/8$ → C_4:$(p-1)(q^2-1)/8$ = $(2k+1-1)\ [(2h+1)^2-1]/8$** **W : Conversion phase by substitution with p=2k+1 and q=2h+1** **D_5: C_4 → C_5:$(p-1)(q^2-1)/8$ = $2k*(4h^2+4h)/8$** **W : calculus rules** **D_6: C_5 is not equal to 2 times something → C_6: I cannot conclude that $(p-1)(q^2-1)/8$ is an even number** **W : even number property** *In proof Elio performs other transformations of the formula but without any result. The other arguments are similar to the previous and they carry out the same result: Elio is not able to construct the proof.*

Elio has solved the problem but he is not able to construct the proof. The strength of the deductive chain seems to be so strong that Elio is not able to construct *continuity in the referential system* with the argumentation; he manipulates the formula to find an expression of the form "2 times something". He looses the connection with the referential system. We can observe that in this case, structurant argumentation does not produce the connection between arithmetic and algebra; this structurant argumentation is still based on numerical examples. Moreover, there is not an explicit abductive step in structurant argumentation which could help Elio to focus which elements lack to justify conjecture and construct the proof.

CONCLUSION

By means of Toulmin's model, we have analysed two resolution processes of an open problem in algebra. In both cases a structurant argumentation was present. In the first case this argumentation allows the construction of the proof while in the second one this argumentation has not been successfully for the construction of the proof.

At this stage of the research we can conclude that, in solving an open problem in algebra, a structurant argumentation can be useful for the construction of the proof if it favours the continuity between constructive argumentation and proof in the referential system. Moreover, in opposition to the geometrical case, abductive structure in structurant argumentation does not represent one of the possible trouble met by student in the construction of the proof because the strength of deductive structure in algebraic proof prevents at least partially the occurrence of structural continuity between argumentation and proof.

REFERENCES

Arsac, Germain & Mante, (1991) Problème ouvert et situation-problème. *Publication de l'IREM* de Lyon

Arzarello, F., Bazzini, L., Chiappini, G. (1994): *L'algebra come strumento di pensiero. Analisi teorica e considerazioni didattiche.* CNR Genova

Boero, P., Garuti, R., Mariotti M. A. (1996): Some dynamic mental processes underlying producing and proving conjectures. *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education PME-XX*, vol. 2, (pp. 121 – 128), Valencia.

Chevallard Y. (1989) : Arithmetique, Algebre, Modelisation, *Publication de l'IREM* d'Aix-Marseille.

Duval, R. (2002): L'apprentissage de l'algèbre et le problème cognitif de la désignation des objects. *Actes des Séminaires SFIDA-13 à SFIDA-16* Vol. IV IREM de Nice.

Drohuard J.P. (1992) : Shells, shellettes and free shells, a framework for algebraic skills, Psychology for Mathematics Education PMEXVII, Tsukuba, Japan

Garuti, R.; Boero, P.; Lemut, E. & Mariotti, M. A. (1996) : Challenging the traditional school approach to theorems:… *Proceedings of the International Group for the Psychology of Mathematics Education PME-X*X, Valencia, vol. 2, pp. 113-120.

Garuti, R.; Boero, P.; Lemut, E. (1998): Cognitive unity of theorems and difficulties of proof *Proceedings of the International Group for the Psychology of Mathematics Education PME-XXII*, vol. 2, pp. 345-352.

Mariotti, M. A. (2001) Justifying and proving in the Cabri environment, *International Journal of Computer for Mathematical Learning*, Dordrecht: Kluwer.

Pedemonte B. (2002): *Etude didactique et cognitive des rapports de l'argumentation et de la démonstration en mathématiques*, Thèse de Doctorat, Université Joseph Fourier, Grenoble.

Pedemonte B. (2005): Quelques outils pour l'analyse cognitive du rapport entre argumentation et démonstration *Recherche en didactique des mathématiques* 25, 3, 313-348

Pedemonte B. (in print): How can the relationship Between Argumentation and Proof be analysed? *Educational Studies in Mathematics*

Peirce C. S. (1960): *Collected papers* Cambridge, M A: Harvard University Press.

Polya G. (1962): *How to solve it?* Princeton University Press, New York.

Toulmin S. E. (1993): *The use of arguments.* Cambridge: University Press, 1958; Traduction française par De Brabanter P. *Les usages de l'argumentation*, Presse Universitaire de France.

MATHEMATICAL PROOF: TEACHERS' BELIEFS AND PRACTICES

Antonis Sergis

University of Bristol, Graduate School of Education, UK

The gap between the mathematical curriculum and what is actually taught in classrooms is an educational worry that requires closer investigation. Teachers' beliefs can possibly throw some light on the reasons explaining this gap. This paper discusses results of my masters' research, illustrating how teachers' beliefs play out in their practices and focuses on the ways these conceptions influence, particularly, the teaching of mathematical proof. The paper aims to point out two teachers' different views of what constitutes proof and the functions of proof they chose to integrate into their teaching practices. Finally, this research sketches some educational implications to improve teachers' - and consequently students' – performances in relation to proof in mathematics.

INTRODUCTION

Teachers' beliefs play a fundamental role in effective mathematics teaching. Most researchers in the area have examined primary school or pre-service teachers' beliefs and practices (Foss & Kleinsasser, 1996; Thompson, 1992; Ernest, 1988; Hanna, 1989). Yet, few examples can be found in the literature about specific subject matter knowledge and beliefs (Ball, 1990; Even, 1993; Tirosh & Graeber, 1990) and fewer about beliefs and proof (Jones, 1997; Hoyles, & Küchemann, 2002). This study explores the relationship between beliefs and proof in the context of secondary school mathematics.

The general motivation for this study derives from my need to call into question the idea that "teachers teach the way they have been taught" (Frank, 1990, p. 12). Mathematical research (Pepin, 1999; Knowles, 1992, Borko, Flory & Cumbo, 1993) has shown that teachers' beliefs are formed during their schooling years, are shaped by their experiences as pupils and hardly change. Furthermore, teachers' conceptions and feelings are revealed during their lessons and affect their decision-making (Woods, 1996), goals (Nespor, 1987), task-defining (Pajares, 1992), priorities (Aguirre & Speer, 2000) and their overall pedagogical approach. As a result, it is questionable whether all students are taught the same mathematics. The students' knowledge and skills are dependent on teachers' beliefs of the mathematical content to be taught. The challenge for the educational community is to provide appropriate training to teachers in order to help them reflect and control the influence of their personal conceptions of mathematics on their practices.

The paper is divided into three sections: in *Section 1* the theoretical framework is discussed; in *Section 2* research tools and methodology are explained; in *Section 3* an analysis of results is presented and, finally, some conclusions are sketched at the end.

1. THEORETICAL BACKGROUND

Teachers' beliefs

Beliefs are defined as conceptions, personal ideologies, world views and values that shape practice and orient knowledge (Ernest, 1989; Thompson, 1992). Teachers often resist adopting educational changes because "changing beliefs causes feelings of discomfort, disbelief, distrust and frustration" (Anderson & Piazza, 1996, p. 53). Nevertheless, recent researchers (Kagan, 1992; Franke *et al*., 1998) argue that lasting changes may occur if teachers try new strategies in their classrooms and reflect on their own belief systems. However, according to Richardson (1996, p. 114) "it cannot be assumed that all changes in beliefs translate into changes in practices".

The relationship between teachers' beliefs and practices is complex. Pepin (1999) found that teachers' conceptions of mathematics and its teaching and learning are not related in a simple cause-and-effect way to their instructional practices. Foss & Kleinsasser (1996) described this relationship as symbiotic; Cohen (1990) identified inconsistencies between teachers' professed beliefs and teaching. The key issue is to find ways to increase teachers' awareness of their beliefs, conceptions and ideas about mathematics. This paper focuses on a specific mathematical aspect, the proving process.

Mathematical proof

Proof can be defined as "ways of convincing someone else of the truth of a statement" (Gutierrez & Jaime, 1994, p. 3). Students often have poor performance and understanding in mathematical proof. According to Schoenfeld (1994, p. 75), "in most instructional contexts proof has no personal meaning or explanatory power for students". Also "students judge that after giving some examples which verify a conjecture they have proved it" (Hoyles, 1997, p. 7). Many of the students' difficulties are due to confusions resulting from their teacher's approaches to proof. Ernest (1988), among others (Thompson, 1984; Calderhead, 1996; Cohen, 1990), claims that teachers' performance is highly depended on their system of beliefs. Therefore, it is vital to examine what kind of conceptions of mathematical proof and knowledge teachers hold because, as Jones (1997, p. 16) states, "the successful teaching of mathematical proof depends crucially on the subject knowledge of mathematics teachers".

2. RESEARCH TOOLS AND METHODOLOGY

This study was carried out in Bristol, UK. Two secondary teachers – George and Nicky – selected purposely, were observed carrying out two lessons each and were interviewed based on pre-observational tasks (concept map and proving task[1]). The

[1] Retrieved from the "Longitudinal Study of Mathematical Reasoning" (1999-2003) project (Year 8 activities) and modified.

tasks encouraged them to talk about their ideas, understanding and conceptions about the *nature* and the *function* of proof in the school context. Lesson observations showed the ways in which those beliefs were carried out, in respect to the tasks set and the questions asked by the teachers.

More specifically, the teachers firstly drew a concept map each to show their understanding of the nature of proof. Also, they completed a proving test, solving problems and providing marks for different sets of responses. This activity provided information about their teaching approach. Afterwards, I interviewed them based on those tasks to reveal their personal constructs of proof. The second phase included two observations of lessons about probabilities with Year 8 students for each teacher. The four lessons included activities with coins and dice and the possible outcomes, for example the number of heads and tails with 2 or 3 coins; the number of 5's and 6's with 2 or 3 dice etc. The observations gave a sample of teachers' instructional approach and behaviour in the classroom. Finally, the teachers reflected, commenting on those lessons. This process provided deeper insight into their beliefs' systems.

The research questions of the study were: (1) What are teachers' conceptions about the nature and role of proof in the context of secondary school mathematics? (2) What is the relationship between teachers' conceptions of proof and their practices?

In this paper I will try to explore only some aspects of these questions and show their relevance for the teaching of mathematical proof.

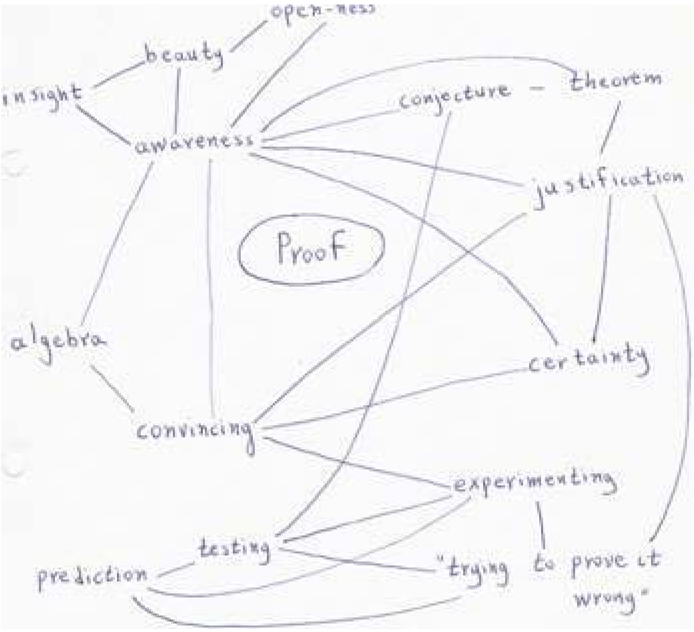
3. FINDINGS AND ANALYSIS

To answer the research questions I designed a theoretical framework which identified teachers' beliefs according to their responses. Particularly, the analysis is based on the different functions of proof: verification (Bell, 1976), explanation (Hersh, 1993), communication (Raman, 2003), discovery (Schoenfeld, 1986) and systematization (Knuth, 2002). Furthermore, the data analysis provided five clusters which allowed comparisons between the two case studies: a) beliefs about the nature of proof; b) beliefs about the functions of proof; c) discussions and group work; d) formal and semi-formal teaching approach to proof and e) classroom culture.

3.1. Case study #1: George

George is the Head of Mathematics. He has ten years of experience and he has got a Master's Degree in Mathematics Education. George is currently working on a PhD proposal. I observed him teaching two lessons with Year 8 students about probabilities. These lessons provided a snapshot of his practices in respect to proof. In this section I summarize his approach and beliefs about proof derived from the concept map, the proving task and the lessons.

Concept map



George used 14 key words in total to draw his concept map (see diagram 1 below). He started from the word "awareness" which he considers to be essential for proof. He links this key word with 9 other words related to proof – this is more than any other word used in the map. He continued with "open-ness" and then "beauty" and linked those two to "awareness". At that point, he added insight which also linked to "beauty" and "awareness". Then the other words followed. The word "proof" is in the center and there are not any links to it at all. Four out of thirteen key words are in *–ing* format (convincing, testing, experimenting and "trying to prove it wrong") and which he relates to classroom activities and context. George's vocabulary during the discussion about his concept map included other phrases or terms such as : incredible pleasure, I always encourage people, motivation to try to prove all cases, community of mathematicians, less rigorous, convince the community.

Diagram 1: George's concept map

Proving task

George finds Ben's answer the best one because it is the only one which provides awareness and explains "why":

> "I suppose it seems to capture the essence of "why", it seems that he got the key awareness of why he got 27 and he seem to describe that awareness very clearly. I didn't have an awareness why it was 27 when I was reading it and when I came to that I said: Oh yes!OK. So it promotes awareness in me about this problem which I suppose to me is what the best groups do."

He gave Ben the mark 10 out of 10 because "he had justified in terms of mathematical structure" and that is what the National Curriculum sets in the marking criteria. George admits that Ben's statement is not an axiomatic proof or absolutely rigorous but it is still a convincing proof in terms of communication. Amina got 4/10 because "she does not seem to have awareness of the problem and probably she has only convinced herself but not the others". Carol (1/10) and Davor (2/10) "are in a lower level of understanding the problem".

Lessons

Commenting on the lessons he had, George feels that he had useful conversations and that probability games helped students to change and learn.

> "Proof for me is not a separate mathematical activity, so I don't think I ever set out to teach proof as such […] if there is something interesting going on in my lesson then proof will be around I'm sure. so even for example doing probability it felt as so what we were discussing with the two coins was an aspect of proof: How can we be sure that the analysis into 1/4 , ¼ and ½ is the correct one? so to me proof is about convincing myself convincing others and as a class coming to some agreement about what we think is the case…and that really for me is what proof is […]"

He believes algebra is essential and important to proof because it answers many "whys" and this is the kind of classroom environment he tries to create. His instruction promotes understanding and the answering of students' questions, "why?"s. This was clear when I observed the language and the approach he uses in his lessons (*italics added*):

> "What do you mean by..? We need to assign in theory-theoretical probability-how these are likely to happen? Can you say *why*? Can anybody help us sort out the 26/52?"

> "Can anybody *apply* Jamie's idea in two dice? Is the Tail-Head same with Head-Tail? What actually happened when you played the game?[…] There is a *need* now (*at he end of the discussion*) to hear what people think..whether we need to have H-T or T-H probabilities"

> "Write a *prediction* about which column will win […]"

> "Your *challenge* is what is the probability for the other games…I suggest to start with 3 coins…What do you *expect* to see in these 4 columns …could you make some *predictions* with 4 dice and 5 dice?..Tom noticed that as we go from 2 dice to 3 dice we double the possibilities. Maybe we could find some *patterns* […]"

George explains that alongside proving, making conjectures and theorems, algebraic process is integrated in his lessons. This is consistent with the example he remembered during the interview after the second lesson:

> "[…] and we worked with that (activity) for 6 lessons and I showed them how to prove in algebra, about why this must work and I think this was a powerful lesson for them. It was their first instruction in secondary school to algebra and it was very complex but it was answering the questions they had […]".

> "[…] when you try to make algebraic statements the question is always around about how can we be sure this is always the case?".

George also speaks about mathematical community. He obviously prefers the *proof that explains* to the *proof that proves* (Hanna, 2000). He talks about proof as explanation and communication:

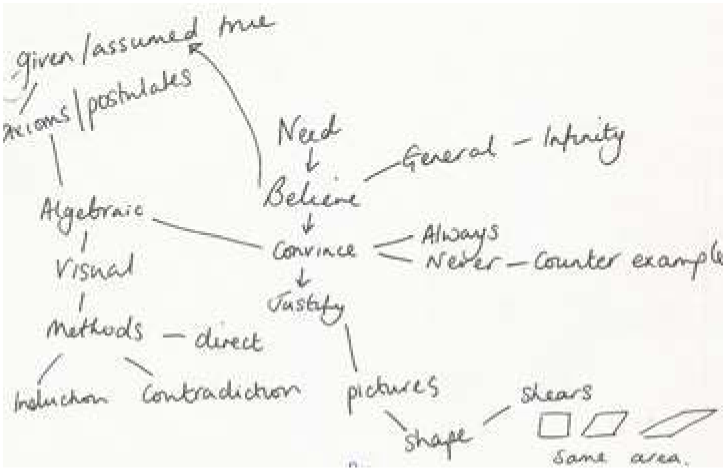
> "[…] I believe that the test of proof is 'Does it convince the community?'"

"[…] the mathematical world has set a very high standard of what form this might need to take. In the classroom context it might be less rigorous but is still the issue 'Does this proof convince the classroom?'".

3.2. Case study #2: Nicky

Nicky is a less experienced teacher having taught mathematics for two and a half years. She is currently studying for her Masters in Mathematics Education. I observed her doing two lessons with Year 8 students (a different group from George's Year 8 class) about probabilities. These lessons were the same as George's.

Concept map



Nicky produced a list of 23 key words and she used 21 words to draw her concept map (diagram 2) although not all were the same with those in the list. Actually, while she described how she drew the concept map she asked herself questions at the same time like: "What ways are there to justify things? How do I convince myself? etc.". Her central words are clearly "believe" and "need". She argues that the core function of proof is always to convince yourself and others, even though you can use different methods to explain why a statement is true or not.

Diagram 2: Nicky's concept map

She uses the word "algebraic" in relation to her personal experiences:

"[…] often I use algebra to convince myself of something if I am working on maths on my level".

However, algebraic proof is not necessarily the only way to convince yourself or the others about the truth of a statement. She believes that there are more ways to be convinced such as visual images and diagrams.

Proving task

Nicky thinks Ben's statement is a good proof because it is general:

"He talked about all cases, so he has talked about everything, so this is general, he is giving a general argument about why this got to be 27, so he is convincing."

She marked this answer with 10/10 because Ben "shows he understands what it means to prove" and because "his method would work for any numbers whereas the other methods would not”. Consequently, Nicky gave 8/10 to Amina because “she only tried some examples and not all the different ways” and convinced herself but not everybody else. Amina could not find a counter example and her method was exhaustive but she got the second highest mark because she has done considerably more work on the problem than Carol (1/10) and Davor (4/10) who did not understand the problem.

Lessons

Nicky feels that the probability game easily convinced her students and she was happy that she did not have to spend much time on that. The whole classroom discussions were fruitful and everyone was involved, asking intelligent questions. She liked that some clear explanations came from her students and she usually wrote on the board any conjecture:

> “Look at the graph. Do you think this is a good picture of the probabilities? Why this bar is bigger than the others?”

> “This is Carly’s *conjecture*: If you had two 8-sided dice would there be 64 possible outcomes from adding the totals together? Would a 9-sided dice have the same *pattern*/ graph?”

> “There are 6 ways to get 7’s so I *expect* the bar to be bigger. The more times you play the more times you expect to get a triangle. This is your *theoretical probability* and you expect this pattern […]”

> “There are 15 ways to get 2 6’s. Is anyone not *convinced* of what she said? She *proved* her answer.”

She is trying to create a culture where all of the students want to prove their conjectures and convince themselves and everyone else in the class:

> “[…] what usually happens is somebody disprove it by giving a counter example or prove it by giving a very *clear explanation and convincing everybody else* in the class […] I tend to talk about proof in the context of their *own conjectures* they come up with […] I hope that there is always a space in the class to prove whatever problem is.”

Informal methods “may help students develop an inner compulsion to understand why a conjecture is true” (Hoyles, 1997, p. 8). Therefore, compared to George’s conceptions, she seems more detached from the formal idea of proof in the school context. Like many teachers (see Martin & Harel, 1989) her description of formal proofs is very ritualistic in nature, tied to prescribed formats and the use of particular language. Nicky also uses the word “need” which shows that justification in terms of personal convincing is the primary function of proof for her.

3.3. Comparison between George and Nicky

Vollrath (1994) claims that judgments by teachers influence students’ appreciation of a theorem. George’s response to proof is *affective* (beautiful, surprising, interesting) and Nicky’s is *cognitive* (special case, inference) and obviously their explicitly and implicitly expressed views affect students' reasoning skills. Teachers should be aware of their mathematical language and "try to balance the different aspects of knowledge, usage, beauty, culture" (Vollrath, 1994, p. 360).

Both teachers set, as major priorities, classroom discussions and questioning. Furinghetti & Olivero (2001) underline the value of collaborative work and indicate the need for children to share, compare and exchange ideas through discussions. Also, Balacheff (1999) argues that the classroom as a scientific community can be an effective way of making room for proof in school mathematics. Such a classroom environment encourages deductive reasoning. On one hand, Nicky provides her students with experiences of more informal methods of proof and opportunities to formulate and investigate conjectures. On the other hand, George wants his students to be "always wanting to know why something works and have an interest in trying to prove it". He asks interesting questions that lead students to make and prove conjectures so he claims that his students produce the proofs. However, according to Herbst (2002, p. 198), if students fail to come up with the statement of a conjecture the teacher would have doubts whether this is due to their lack of reasoning skills or the teachers' failure to provide a fair task. Nicky and George work with experimenting, conjecturing and testing in their lessons – elements necessary to create a classroom culture where proof is always involved.

Results showed that teachers' existing conceptions of proof have some consistencies and some inconsistencies with their practices. Both teachers hold similar beliefs about the core function of proof – "convince myself and others" – although they are different in the ways each of them reach conviction. George seems more dedicated to the formal and public aspect of proof in his class, whereas Nicky accepts several forms of justification which can satisfy pupils; personal doubts about the truth of a statement (see diagram 3 below).

	Nicky	George	
Personal aspect			Public aspect
Verification			Communication
Justification			Explanation
Informal approaches			Formal approaches
Empirical			Abstract

Diagram 3: Comparison between teachers' beliefs of proof

In other words, George can be characterized as serving proof in the function of communication and explanation and Nicky as serving proof in the function of justification and verification. Chen & Lin (2002) would characterize Nicky's pedagogical views about teaching proof as a mixture of *convincing-formal view* and *instructional explanatory view*. This means that a teacher convinces students of the truth by manipulation, special cases and demonstrates some kind of explanation. George has a *discursive explanatory view* where the explanation results from students' discourse.

4. EDUCATIONAL IMPLICATIONS AND CONCLUSIONS

Knuth (2002) suggests that implementing "proof for all" might be difficult for teachers. Teacher training programmes and curriculum planners should prepare teachers to teach mathematical proof in the school context, bearing in mind three important elements: a) the levels of proving; b) the functions of proof and c) the approaches to proof.

The comparison between the two teachers reveals the issue of the taught curriculum. Obviously the national guidelines about proof are the same for all teachers; however students do not receive the same instruction. For example, George and Nicky have different teaching approaches based on their beliefs about what proof is. There are also some other factors which influence their performance such as their content knowledge, students' attainment levels, the school and classroom environment and the social context. Consequently, the same material is taught differently and students do not gain the same understanding of the concept of proof.

In conclusion, I highlight the fact that the stronger influence on the relationship between teachers' beliefs and practices does not derive from their past lives and the ways they have been taught; it is teachers' engagement in practical inquiry and their experiences in the classroom (Franke *et al*., 1998) which forms their teaching. Therefore, teachers need to select the most successful methods and prepare effective tasks that respond to the demands of the students and promote *mathematical enculturation* (Bishop, 1988).

REFERENCES

Aguirre, J. & Speer, M. N. (2000): Examining the relationship between beliefs and goals in teaching practice. *Journal of Mathematical Behavior* 18 (3), 327-356.

Anderson, D. S. & Piazza, J. A. (1996): Teaching and learning mathematics in constructivist pre-service classrooms. *Action in Teacher Education* 18 (2), 51-62.

Ball, D. L. (1990): Examining the subject-matter knowledge of prospective mathematics teachers. *Journal for Research in Mathematics Education* 21 (2), 132-143.

Balacheff, N. (1999): Contract and custom: Two registers of didactical interactions. *The Mathematics Educator* 9, 23-29.

Bell, A. (1976): A study of pupils' proof-explanations in mathematical situations. *Educational Studies in Mathematics* 7, 23-40.

Bishop, A. J. (1988): *Mathematical Enculturation*. Dordrecht: Kluwer.

Borko, H., Flory, M. & Cumbo, K. (1993): Teachers' ideas and practices about assessment and instruction: A case study of the effects of alternative assessment in instruction, student learning and accountability practices. *Proceedings of the Annual Conference of the American Educational Research Association, Atlanta.*

Calderhead, J. (1996): Teachers: Beliefs and knowledge. In Berliner, C. D. & Calfee, C. R. (Eds.), *Handbook of Educational Psychology*, 709-725. New York: Macmillan.

Chen, I. & Lin, F. (2002): The evolution of a student teachers' pedagogical views about teaching mathematics proof. *Proceedings of the XVI Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, 225-232.

Cohen, D. (1990): A revolution in one classroom: The case of Mrs. Oublier. *Education, Evaluation and Policy Analysis* 12 (3), 311-329.

Ernest, P. (1988): The impact of beliefs on the teaching of mathematics. *Paper presented for ICME VI, Budapest, Hungary.* Retrieved August, 15, 2006 from http://www.people.ex.ac.uk/PErnest/impact.htm

Ernest, P. (1989): The knowledge, beliefs and attitudes of the mathematics teacher: A model. Journal of Education for Teaching 15 (1), 13-33.

Even, R. (1993): Subject-matter knowledge and pedagogical content knowledge: prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education* 24 (2), 94-116.

Foss, H. D. & Kleinsasser, C. R. (1996): Pre-service elementary teachers' views of pedagogical and mathematical content knowledge. *Teaching and Teacher Education* 12 (4), 429-442.

Frank, M. L. (1990): What myths about mathematics are held and conveyed by teachers? *The Arithmetic Teacher* 37 (5), 10-12.

Franke, M., Fennema, E., Carpenter, T., Ansell, E. & Behrend, J. (1998): Understanding teachers' self-sustaining, change in the context of professional development. *Teaching and Teacher Education* 14, 67-80.

Furinghetti, F. & Olivero, F. (2001): Students approaching proof through conjectures: snapshots in a classroom. *International Journal of Mathematical Education in Science and Technology* 32 (3), 319-335.

Gutierrez, A. & Jaime, A. (1994): A model of test design to assess the Van Hiele levels. *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, 41-48.

Hanna, G. (1989): More than formal proof. *For the Learning of Mathematics* 9 (1), 20-23.

Hanna, G. (2000): Proof, explanation and exploration: An overview. *Educational Studies in Mathematics* 44, 5-23.

Harel, G. & Sowder, L. (1998): Students' proof schemes: Results from exploratory studies. In A. Schoenfeld, J. Kaput & E. Dubinsky (Eds.) *Research in collegiate mathematics education* III (234-283). Washington, DC: Mathematical Association of America.

Herbst, G. P. (2002): Engaging students in proving: A double bind on the teacher. *Journal for Research in Mathematics Education* 33(3), 176-203.

Hersh, R. (1993): Proving is convincing and explaining. *Educational Studies in Mathematics* 24, 389-399.

Hoyles, C. (1997): The curricular shaping of students' approaches to proof. *For the Learning of Mathematics* 17 (1), 7-16.

Hoyles, C. & Küchemann, D. (2002): Year 8 students' proof responses: some preliminary findings from four groups. *Proceedings of the British Society for Research into Learning Mathematics* 20 (3), 43-48.

Jones, K. (1997): Student-teachers' conceptions of mathematical proof. *Mathematics Education Review* 9, 16-24.

Kagan, D. M. (1992): Professional growth among pre-service and beginning teachers. *Review of Educational Research* 62 (2), 129-169.

Knowles, J. G. (1992): Models for understanding pre-service and beginning teachers' biographies: Illustrations from case studies. In Goodson I. F. (Ed.), *Studying teachers' lives*. London: Routledge.

Knuth, E. (2002): Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education* 33 (5), 379-405.

Longitudinal Study of Mathematical Reasoning: Student development and School influences (1999-2003). *Retrieved from* http://www.ioe.ac.uk/proof/index.html

Martin, W. G. & Harel, G. (1989): Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education* 20 (1), 41-51.

Nespor, J. (1987): The role of beliefs in the practice of teaching. *Journal of Curriculum Studies* 19 (4), 317-328.

Pajares, M. F. (1992): Teachers' beliefs and Educational Research: Cleaning up a messy construct. *Review of Educational Research* 62 (3), 307-332.

Pepin, B. (1999): *Epistemologies, beliefs and conceptions of mathematics teaching and learning: the theory and what is manifested in mathematics teachers' work in England, France and Germany*. The Open University, UK, 2(1), 127-146. TNTEE Publications.

Raman, M. (2003): Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics* 52, 319-325.

Richardson, V. (1996): The role of attitudes and beliefs in learning to teach. In J. Sikula (Ed.), *The Handbook of Research in Teacher Education*, 102-119. New York: Macmillan.

Schoenfeld, A. H. (1986): On having and using geometric knowledge. In Hiebert, J. (Ed.), *Conceptual and Procedural Knowledge: The case of Mathematics*. Hillsdale: Lawrence Erlbaum.

Schoenfeld, A. H. (1994): What do we know about mathematics curricula? *Journal of Mathematical Behavior* 13 (1), 55-80.

Thompson, A. (1984): The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics* 15, 105-127.

Thompson, A. (1992): Teachers' beliefs and conceptions: A synthesis of research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*, 127-146. New York: Macmillan.

Tirosh, D. & Graeber, A. (1990): Inconsistencies in pre-service teachers' beliefs about multiplication and division. *Focus on Learning Problems in Mathematics* 12, 65-74.

Vollrath, H. J. (1994): On the appreciation of theorems by students and teachers. In Robitaille D. F., Wheeler D. H., Kieran C. (Eds.) *Selected lectures from the 7th International Congress in Mathematical Education* (pp. 353-365). Saint-Foy: Les Presses de l' Universiti Laval.

Woods, D. (1996): *Teacher cognition in language teaching: Beliefs, decision-making and classroom practice*. Cambridge: Cambridge University Press.

THE MENTAL MODELS THEORY OF DEDUCTIVE REASONING: IMPLICATIONS FOR PROOF INSTRUCTION[1]

Andreas J. Stylianides | Gabriel J. Stylianides
University of Oxford, U.K. | University of Pittsburgh, U.S.A.

There are currently increased efforts to make proof central to school mathematics throughout the grades. Yet, realizing this goal is not easy, as it requires that students master several abilities. In this article, we focus on one such ability, namely, the ability for deductive reasoning. We first offer a conceptualization of proof, which we use to delineate our focus on deductive reasoning. We then review Johnson-Laird's mental models theory – a well-respected psychological theory of deductive reasoning – in order to enhance what is currently known in mathematics education research about deductive reasoning in the context of proof.

INTRODUCTION

There are currently increased efforts to make *proof* central to school mathematics throughout the grades (e.g., Ball & Bass, 2003; NCTM, 2000). Yet, realizing this goal is not easy, as successful engagement with proof requires that students master several abilities, such as the ability to recognize the need for a proof (e.g., Boero et al., 1996; Mason et al., 1982), the ability to understand the role of definitions in the development of a proof (e.g., Mariotti & Fischbein, 1997; Zaslavsky & Shir, 2005), and the ability to use deductive reasoning (e.g., Foltz et al., 1995; Polya, 1954). In this article, we focus on the ability for *deductive reasoning*.

Available mathematics education research on proof offers: (1) existing evidence that, and insights into how, supportive classroom environments can enable even elementary school students to use deductive reasoning to construct arguments and proofs (e.g., Ball & Bass, 2003; Maher & Martino, 1996); (2) understanding of common difficulties that students face in using deductive reasoning in the context of proof (e.g., Coe & Ruthven, 1994; Hoyles & Küchemann, 2002); and (3) understanding of social and cognitive factors that play a role in students' ability to use deductive reasoning in the context of proof (e.g., Balacheff, 1991; Boero et al., 1996). The findings of this research can be complemented by the findings of psychological research on cognitively guided ways to enhance acquisition of this ability, such as the research associated with Johnson-Laird's (1983; Johnson-Laird & Byrne, 1991) *mental models theory*. Although this psychological research makes only few connections to the notion of proof, it can offer useful insights into proof instruction in school mathematics. Of course, incorporating findings of psychological studies on deductive reasoning into useful practices for promoting students' ability for proof requires first a considerable amount of interdisciplinary research to build necessary bridges between psychology and mathematics education.

Our objective is to review Johnson-Laird's mental models theory – a well-respected psychological theory of deductive reasoning – in order to enhance what is currently known in mathematics education research about this ability in the context of proof. The article is structured into two sections. In the first section, we offer a conceptualization of the meaning of proof in school mathematics and we use this conceptualization to delineate our focus on deductive reasoning. In the second section, we review the mental models theory of deductive reasoning and we discuss implications of this theory for proof instruction.

PROOF AND DEDUCTIVE REASONING

Our conceptualization of the notion of proof in this article is summarized as follows:

> *Proof* is a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:
> 1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
> 2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
> 3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community. (Stylianides, 2007, p. 291)

The conceptualization of proof breaks down each mathematical argument into three major *components*: the set of accepted statements (e.g., definitions, axioms, theorems), the modes of argumentation (e.g., application of logical rules of inference like modus ponens), and the modes of argument representation (e.g., verbal, pictorial, algebraic). The use of the terms "true," "valid," and "appropriate" in the conceptualization should be understood in the context of what is typically agreed upon in the field of mathematics nowadays. Of course, this is not to say that these terms have universal meaning in the field of mathematics nowadays, but it is beyond the scope of this article to elaborate on this issue.

The notion of deductive reasoning corresponds to the component *modes of argumentation*. The conceptualization denotes that such modes used in an argument that qualifies as a proof need to be valid and, therefore, they need to support logically necessary inferences from a given set of premises. According to commonly accepted notions of deductive reasoning, logically necessary inferences directly implicate the use of deductive reasoning. For example, Klaczynski and Narasimham (1998) note that deductive reasoning refers to logically necessary inferences drawn "from a general set of givens or premises" (p. 865). Important to note is that we do not associate the notion of deductive reasoning with particular modes of representation, such as modes that may be characterized as formal versus informal.

MENTAL MODELS THEORY AND INSTRUCTIONAL IMPLICATIONS

Before we describe the mental models theory, three caveats are in order. First, although the mental models theory is well respected in cognitive psychology, by choosing it for presentation in this article we do not suggest that it is the best theory currently available in the domain of deductive reasoning. For example, there is Rips's (1994) *theory of deductive reasoning*, which is based on natural-deduction systems for predicate logic and has levels of empirical support similar to the mental models theory. Second, the mental models theory provides an explanation of reasoners' thinking processes on a small range of deductive reasoning tasks, namely, syllogistic inference tasks. Other theories address different kinds of deductive reasoning tasks. For example, the *pragmatic reasoning schema theory* (Cheng & Holyoak, 1985) addresses selection tasks. Third, there is still much to be learned about the mental models theory and how it relates to other theories of deductive reasoning. For example, although we have comparisons of the mental models theory and the pragmatic reasoning schema theory (e.g., Moshman, 1998), psychologists have not yet analyzed fully the relationship between these two theories.

To conclude, our discussion of the mental models theory is intended to initiate discussions and interdisciplinary efforts on how proof instruction can benefit from, and use the findings of, psychological theories of deductive reasoning. We see our discussion as the very first step in a long process that will consider other theories besides the one considered in this article.

Presentation of the theory

The mental models theory assumes that deductive reasoning, as it applies to syllogisms (i.e., arguments from premises to an inference or a conclusion), depends on three main stages (Johnson-Laird & Bara, 1984). First, the reasoner constructs a mental model of the information presented in the premises of a syllogism, where by "mental model" is meant a representation in the mind that has a structure analogous to the structure of the situation it represents. Second, the reasoner scans this model for an informative conclusion that is true. Third, the reasoner searches for alternative mental models that may lead to refutation of the conclusion (counterexamples). In this approach, developmental changes in the ability for deductive reasoning reflect: (1) improvement of the linguistic competence to comprehend logical terms (e.g., and, or, not, if, none, some, all) in the premises and, thus, of the ability to construct appropriate models of those premises; and (2) advancement in the management of these models due to increase in processing capacity (Johnson-Laird, 1990).

Johnson-Laird and Byrne (1991) argue that "people make deductions by building models and searching for counterexamples" (p. 203). They consider that the ability for deductive reasoning is equivalent to the "capacity to build models of the world, either directly by perception or indirectly by understanding language, and [the] capacity to search for alternative models" (p. 204). According to the mental models theory, the unfolding of these capabilities occurs under the control of innate constraints:

> What develops in childhood is the ability to understand language, the processing capacity of *working memory* (Hitch and Halliday, 1983; Case, 1985), and the meta-ability to reflect on one's own performance. Seven year-olds cannot cope with syllogisms because

> they do not understand quantifiers correctly (see Inhelder and Piaget, 1964). Nine year-olds can cope with one-model syllogisms, but not with more than one model (Johnson-Laird, Oakhill, and Bull, 1986; Acredolo and Horobin, 1987). Their *working memory* appears to lack sufficient capacity to retain alternative models of the premises. (Johnson-Laird & Byrne, 1991, p. 204; emphasis added)

As the excerpt above suggests, the degree of success with which mental model construction and examination can be achieved depends on a person's working memory capacity. In other words, the number of models that are constructed and the figural arrangement of terms that can be made within the premises[2], which constitute the two major factors that determine the difficulty of making inferences, seem to be intimately related to working memory:

> The effects of both number of models and figure arise from an inevitable bottleneck in the inferential machinery: the processing capacity of working memory, which must hold one representation in a store, while at the same time the relevant information from the current premise is substituted in it. (Johnson-Laird, 1983, p. 115)

In general, working memory capacity plays a central role in the theory's successful accounting for patterns of performance in deductive reasoning (Johnson-Laird, 1983). More specifically, errors occur because limitations in working memory capacity make people fail to consider all possible models of the premises that would provide them with counterexamples to the conclusions they derive from their initial models (Johnson-Laird, 1983; Johnson-Laird & Byrne, 1991). In turn, this limits individuals' ability for validation, for it constrains their ability to consider more than one model at a time (Johnson-Laird & Bara, 1984).

Johnson-Laird and colleagues' (1986) experiments with two groups of children (9- to 10-year-olds and 11- to 12-year-olds) provide support to the claim that the ability to solve syllogistic problems is associated with the number of mental models that have to be constructed for the solution of a given problem. In one experiment – where the two groups of children drew conclusions from 20 pairs of syllogistic premises – no child in either group made a correct response to the three-model problems, whereas all subjects (the only exception being one 9-year-old) made at least one correct response to a one-model problem. In another experiment – where 16 11-year-olds were tested on all 64 possible forms of syllogistic premises[3] – the children made only 2% correct responses to the three-model problems, as compared to 26% and 63% correct responses to the two- and one-model problems, respectively. Overall, in both experiments, "performance was best with one-model problems and better than chance with two-model problems" (Johnson-Laird et al., 1986, p. 52); correct responses with three-model problems were virtually non-existent.

Investigations with adults revealed a similar pattern of performance, that is, best performance on one-model problems and worst on three-model problems, with the main difference being that the number of problems of each type that the adults solved correctly was typically larger than the corresponding number for children.

Interestingly, when adults had only 10 seconds to respond to syllogistic premises, their performance dropped to a level almost identical to that of the 11-year-olds (Johnson-Laird & Bara, 1984). Anderson et al. (1996) make an attempt to explain these results by using Case's (1984) ideas about short-term *operating* and *storage* spaces: "while the overall capacity of short-term memory does not increase as a function of development, the effect of practice at tasks results in more efficient use of short-term *operating* space, leaving greater capacity in short-term *storage* space" (Anderson et al., 1996; p. 270). Therefore, "[a]s learners get older, they become more adept at building, maintaining in memory, and testing a transitory mental model" (Anderson et al., 1996; p. 270). Increases in the information storage capacity would clearly be beneficial for learners' capacity to achieve these processes.

Discussion of the theory

The mental models theory has not been applied in the teaching and learning of proof, so the examples one can find in the literature illustrating the theory are not focusing on the notion of proof (rather, they are mostly syllogistic tasks placed in non-mathematical contexts). We begin our discussion of the theory with an example of a proving task that we constructed in order to illustrate possible applications of the theory in the particular domain of proof.

> Consider the following two premises, which are basically definitions for multiples of 3 and 6:
>
> An integer is a multiple of 3 if and only if it is three times an integer.
>
> An integer is a multiple of 6 if and only if it is six times an integer.
>
> What can be said (if anything) about any multiple of 6 in relation to a multiple of 3? Prove your answer.

Using the first premise, the following algebraic expression is constructed for any multiple of 3: $3l$, where l is an integer. Likewise, using the second premise, the following expression is constructed for any multiple of 6: $6k$, where k is an integer. Using the information in the two premises, the following model is constructed:

Any multiple of 6 is of the form $6k = 3 \cdot (2k) = 3l$, a multiple of 3.

As this model cannot be falsified with an alternative model, the conclusion is considered valid. That is, we have proved that any multiple of 6 is also a multiple of 3. Successful completion of this proving task depends, given the tenets of the mental models theory, on the linguistic competence of the reasoner to comprehend the logical term "if and only if" in the premises and on the reasoner's processing ability to combine the information from the two premises and search for alternative models.

The structure of the proving task we just analyzed was purposefully organized so that there is a clear set of premises from which the reasoner can draw information to construct a mental model. Yet, most of the proving tasks encountered by students and professional mathematicians do not have this clear structure. Typical proving tasks consist only of the question/prompt (in the particular case: "What can be said [if anything] about any multiple of 6 in relation to a multiple of 3? Prove your answer."), leaving it up to the reasoner to select a collection of relevant premises from his or her community's set of accepted statements (cf. our conceptualization of proof) to construct a proof. Accordingly, the mental models theory seems to be useful in accounting for the solution of a proving task once a set of premises has been specified (either by the task itself or by the reasoner). Of course, the reasoner can revise the set of premises by adding or deleting premises in order to end up with a sufficient set of premises for the solution of the proving task. Each time a new set of premises is established, the mental models theory can be reapplied.

The mental models theory denotes that limited working memory capacity constrains students' performance in deductive reasoning tasks of which proving tasks are a proper subset. This implies that mathematics educators can potentially foster the improvement of students' performance in proving tasks in two interrelated ways: (1) by preventing unnecessary usage of students' working memory when they engage with proving tasks, and (2) by helping students develop strategies for effective managing of their working memory.

An example of (1) is for mathematics educators to engage students in "scaffolded" proving tasks (like the one presented earlier) that specify for the students a small set of relevant premises for the solution of the task. By excluding irrelevant premises from students' consideration when engaging with a proving task, students are freed from the memory-consuming effort to combine information from more premises and construct more complicated mental models than they actually need to. In this way, students are facilitated to focus on the logical structure of the proof and the ideas involved in it. Of course, at some point, educators would like students to become able to identify by themselves the relevant premises for the solution of a proving task. Yet, the kinds of scaffolded proving tasks described earlier can be very useful in the early stages of students' engagement with proof, for they can help students develop necessary skills that will support their independent engagement with proof in the future.

An example of (2) is for mathematics educators to help students develop the strategy of *representing the information in the premises in equivalent and easier to manage forms* (from a working memory standpoint). Looking back to our analysis of the proving task at the beginning of our discussion of the theory, we see that the two premises were reformulated to algebraic expressions, which, due to their conciseness, reduce the processing load thereby facilitating the solution of the task. A related strategy that mathematics educators can assist students to develop is *making efficient use of visual records* such as diagrams (see, e.g., Bauer & Johnson-Laird, 1993; English, 1998; Sweller et al., 1998). This strategy can involve not only the use of a visual record when one is not already offered in a proving task but also the use of an existing diagram to practically integrate disparate sources of information and facilitate solution. It is often the case in proving tasks, especially in geometry, that students are offered a diagram (e.g., a geometric figure) and then a set of givens (premises) which, although refer to the diagram, are separated from the diagram. To make sense of the two sources of information, the diagram and the givens, students must mentally integrate them. For example, to derive any meaning from a given, students must read the given, hold it in their working memory, and then search the diagram for the appropriate referents. This mental integration process is clearly cognitively demanding and occupies a large part of their working memory capacity.

The works of Case (1984) and Anderson et al. (1996) we reviewed earlier suggest that *practice* can help students become more skillful in overcoming the limitations of their working memory capacity, thereby pointing to an educational implication for improving students' ability for deductive reasoning. This implication has first been proposed by Johnson-Laird (1983) based on his observations of "spontaneous improvement in [deductive] reasoning ability simply as a consequence of practice (with no feedback)" (p. 124). Adults in Johnson-Laird's experiments who have been tested twice within a week showed a 10% improvement in their performance, without even forewarning that they would be retested (Johnson-Laird & Steedman, 1978).

The idea that practice can play an important role in the development of students' ability for deductive reasoning, and thus in their ability for proof, is not emphasized in mathematics education research on proof. Research studies on teaching practices that have successfully promoted students' ability for proof do not explicitly identify practice as one of the factors that might have contributed to this success. Part of the reason for which there has been little attention to the potential role of practice in proof learning might be that practice has often been associated with secondary aspects of students' engagement with proof, such as the writing of a proof in the two-column form. According to the two-column form, which prevailed in high school geometry courses in the Unites States for almost a century, "the statements of the proof [are placed] in steps in a column occupying the left half of the page, and … the reasons of the statements [are placed] in steps at the right side of the page, with each reason directly opposite its statement" (Shibli, 1932, p. 145). The emphasis on the form made the writing of a proof a ritual procedure that had to be practiced and memorized; as a result, "the substance of proof as a logical and coherent chain of reasoning that guarantees that something must be true became obscured" (Schoenfeld, 1991, p. 325).

Yet, associating practice only with secondary aspects of students' engagement with proof does not do justice to the potential role that practice can play in proof learning as suggested by the psychological research reviewed in this article. An obvious possible use of practice is to help students develop the kinds of strategies for effective managing of their working memory that we described earlier. In addition, we hypothesize that practice can be used to help students internalize the general logical structure of different proof methods, such as proof by contradiction, thus releasing working memory capacity to be spent in the application of these proof methods. For example, consider the proposition: "There is no smallest positive rational number." In a proof of this proposition by contradiction, one would start by assuming the opposite of the proposition and would show that it leads to a logical contradiction:

> We assume the opposite of the statement we wish to prove: "There is a smallest positive rational number, say *y*." Now let $x=y/2$. Then *x* is a positive rational number that is smaller than *y*. But this contradicts our initial assumption that *y* is the *smallest* positive rational number. So we can conclude that the original proposition must be true – "There is no smallest positive rational number."

If a student who has not internalized the logical structure of the proof method by contradiction attempts to apply the method to prove the proposition, this student will experience increased processing load of working memory and will likely face increased difficulties with the proof.

CONCLUSION

In this article, we reviewed an influential and well-respected psychological theory of deductive reasoning, namely, Johnson-Laird's theory of *mental models*, in order to enhance what is currently known in mathematics education research about this ability in the context of proof. Our review offered useful insights into potentially effective instructional practices for fostering students' ability for deductive reasoning in the context of proof. Nevertheless, it is a long way before these insights can find their way to the practices of ordinary teachers.

A major challenge, but also a primary urgency, for researchers concerned with issues of proof instruction is to identify effective ways to synthesize relevant research programs in mathematics education and psychology. This article has made a first step towards this direction by bringing to the attention of mathematics education researchers a rich body of psychological research on deductive reasoning and by identifying important issues that require research attention. An interdisciplinary and collaborative approach to the problem of promoting proof in students' learning of mathematics promises major advancements.

NOTES

1. The two authors had an equal contribution in writing this article.

2. For example, syllogisms that involve two premises with three terms (X, Y, and Z) can occur in one of four figures as shown below:

X – Y	Y – X	X – Y	Y – X
Y – Z	Z – Y	Z – Y	Y – Z

3. The 64 possible forms of syllogistic premises are derived as follows: four quantifier combinations for each of two premises that can occur in one of four figures as explained in endnote 2 (i.e., $64=4^3$).

REFERENCES

Anderson, T., Howe, C., & Tolmie, A. (1996). Interaction and mental models of physics phenomena: Evidence from dialogues between learners. In J. Oakhill, &

A. Garnham (Eds.), *Mental models in cognitive science* (pp. 247-273). Hove, UK: Psychology Press.

Balacheff, N. (1991). The benefits and limits of social interaction: The case of mathematical proof. In A.J. Bishop (Ed.), *Mathematical knowledge: Its growth through teaching* (pp. 175-192). Netherlands: Kluwer Academic Publishers.

Ball, D.L., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W.G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 27-44). Reston, VA: National Council of Teachers of Mathematics.

Bauer, M.I., & Johnson-Laird, P.N. (1993). How diagrams can improve reasoning. *Psychological Science, 4*(6), 372-378.

Boero, P., Garuti, R., & Mariotti, M.A. (1996). Some dynamic mental processes underlying producing and proving conjectures. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 121-128). Valencia, Spain.

Case, R. (1984). The process of stage transition: A neo-Piagetian view. In R.J. Sternberg (Ed.), *Mechanisms of cognitive development* (pp. 19-44). New York: Freeman.

Case, R. (1985). A developmentally based approach to the problem of instructional design. In S.F. Chipman, J.W. Segal, & R. Glaser (Eds.), *Thinking and learning skill: Research and open question* (Vol. 2). Hillsdale, NJ: Laurence Erlbaum Associates.

Cheng, P.W., & Holyoak, K.J. (1985). Pragmatic reasoning schemas. *Cognitive Psychology, 17*, 391-416.

Coe, R., & Ruthven, K. (1994). Proof practices and constructs of advanced mathematics students. *British Educational Research Journal*, *20*, 41-53.

English, L.D. (1998). Children's reasoning in solving relational problems of deduction. *Thinking and Reasoning, 4*(3), 249-281.

Foltz, C., Overton, W.F., & Ricco, R.B. (1995). Proof construction: adolescent development from inductive to deductive problem-solving strategies. *Journal of Experimental Child Psychology, 59*, 179-195.

Hoyles, C., & Küchemann, D. (2002). Students' understanding of logical implication. *Educational Studies in Mathematics, 51*, 193-223.

Johnson-Laird, P.N. (1983). *Mental models: Towards a cognitive science of language, inference, and consciousness.* Cambridge: Cambridge University Press.

Johnson-Laird, P.N. (1990). The development of reasoning ability. In G. Butterworth & P. Bryant (Eds.), *Causes of development* (pp. 85-110). Hillsdale, NJ: Laurence Erlbaum Associates.

Johnson-Laird, P.N., & Bara, B.G. (1984). Syllogistic inference. *Cognition, 16*, 1-62.

Johnson-Laird, P.N., & Byrne, R.M.J. (1991). *Deduction.* Hillsdale, NJ: Laurence Erlbaum Associates.

Johnson-Laird, P.N., & Steedman, M.J. (1978). The psychology of syllogisms. *Cognitive Psychology, 10*, 64-99.

Johnson-Laird, P.N., Oakhill, J.V., & Bull, D. (1986). Children's syllogistic reasoning. *Quarterly Journal of Experimental Psychology, 38A*, 35-58.

Klaczynski, P.A., & Narasimham, G. (1998). Representations as mediations of adolescent deductive reasoning. *Developmental Psychology, 34*(5), 865-881.

Maher, C.A., & Martino, A.M. (1996). The development of the idea of a mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education, 27*, 194-214.

Mariotti, M.A., & Fischbein, E. (1997). Defining in classroom activities. *Educational Studies in Mathematics, 34*, 219-248.

Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically.* London: Addison-Wesley.

Moshman, D. (1998). Cognitive development beyond childhood. In W. Damon (Series ed.) & D. Kuhn & R.S. Siegler (Vol. eds.), *Handbook of child psychology, Vol. 2: Cognition, perception, and language* (5th Ed., pp. 947-978). New York: Wiley.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Polya, G. (1954). *Induction and analogy in mathematics.* Princeton, NJ: Princeton University Press.

Rips, L.J. (1994). *The psychology of proof: Deductive reasoning in human thinking.* Cambridge, MA: MIT Press.

Schoenfeld, A.H. (1991). On mathematics as sense-making: An informal attack on the unfortunate divorce of formal and informal mathematics. In J.F. Voss, D.N. Perkins, & J.W. Segal (Eds.), *Informal reasoning and education* (pp. 311-343). Hillsdale, NJ: Lawrence Erlbaum Associates.

Shibli, J. (1932). *Recent developments in the teaching of geometry.* State College.

Stylianides, A.J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38*(3), 289-321.

Sweller, J., van Merrienboer, J.J.G., & Paas, F.W.C. (1998). Cognitive architecture and instructional design. *Educational Psychology Review, 10*(3), 251-296.

Zaslavsky, O., & Shir, K. (2005). Students' conceptions of a mathematical definition. *Journal for Research in Mathematics Education, 36*(4), 317-346.

REVIEWING TEXTBOOK PROOFS IN CLASS: A STRUGGLE BETWEEN PROOF STRUCTURE, COMPONENTS AND DETAILS

Stine Timmermann

IMFUFA, Department of Sciences, Roskilde University, Denmark

Abstract. During the first year of a university study in mathematics, "dissection" of mathematical proofs occupy a growing part of the course time. In this paper I investigate how we can describe, characterise, analyse and thus understand what is going on during a presentation of a textbook proof in class. The conclusion is that students' misunderstandings and miscommunications between teacher and students may be explained if the analysis separates between the proof structure, the components in the structure and the details of the proof. Excerpts from a presentation of a proof in an analysis course at a Danish university are used to illustrate this point.

INTRODUCTION

A traditionally taught university course is often divided between lectures where the content of the textbook is explained by the lecturer and exercise/problem solving sessions where the lecturer or a teaching assistant works through assigned tasks. Gradually during a university study in mathematics, "dissection" of proofs (investigating and analysing how different parts of the proof function and how they relate) becomes a more and more important activity in the teaching practice[1] and it is for that reason interesting to investigate this activity in detail. In this paper I want to focus on the following question:

- How can we describe and characterise a teacher's presentation of a proof in the classroom and a dialogue with the students?

The work presented in this paper is part of a larger exploratory project concerning mathematics teaching and problem solving at the tertiary level baring the research question: *In what ways does the teaching practice influence the way students approach and solve mathematical tasks?*

The mathematical subject is moderately advanced mathematical analysis (beyond calculus) where the mathematical tasks demand justification and/or a proof of a claim about properties of or relationships between mathematical objects and concepts. At this mathematical level proof dissection is a very important part of the teaching practice and I want to use this paper to illustrate how a proposed theoretical tool can be used to describe, analyse and characterise textbook proofs and the part of the teaching practice that involves proof reviews.

Although it is common knowledge in the mathematical community that understanding a mathematical proof takes more than just verifying each step in the proof (Bourbaki, 1950), classroom practice is more concerned with analysing the details than with synthesizing, "combining parts to a whole" (Dreyfus, 1991). The discussion of how to present mathematical proofs to students is (naturally) still an ongoing topic in educational research concerning proofs, both with respect to analysing and categorizing "normal" teaching styles (Weber, 2004; Hemmi, 2006) and to suggesting alternative ways to teach proofs (Leron, 1983; Balacheff, 1991; Alibert and Thomas, 1991; Legrand, 2001) often emphasising the "conviction part" of the purpose of proving (Harel and Sowder, 1998). The reason for such research is that "*... the key role of proof is the promotion of mathematical understanding, and thus our most important challenge is to find more effective ways of using proof for this purpose.*" (Hanna, 2000, p. 5-6).

Beside research concerning the teaching of proofs, many research studies concern documentation and analysis of students' difficulties with constructing mathematical proofs (Moore, 1994; Dreyfus, 1999; Weber, 2001; Selden and Selden, 2003) or their perceptions of what constitutes a valid proof (Martin and Harel, 1989; Dreyfus, 1999; Healy and Hoyles, 2000; Raman, 2003).

The literature does not however offer a comprehensive framework for analysing and comparing teaching practices (social perspective[2]) in relation to students' proof production processes (individual perspective[2]). I found it necessary and useful to develop a framework that could be used to analyse my data material. The construction I propose is based on data (a "bottom-up" approach) and mathematically grounded, and can, beside being used in the analysis of classroom proof presentations, also provide a tool for analysing students' proof production processes, thus allowing for a way to relate the teaching of proofs to students' proof construction difficulties (the latter feature, though, is not demonstrated in this paper).

THEORETICAL CONSTRUCTION

The important notions in the proposed theoretical construction are the notions of *structure*, *components* and *details*. A structure is composed of interrelated components. The specific details of each component can vary in number and complexity. When talking about a textbook proof the following definition of the structure, components and details is suggested:

The structure of a proof is a hierarchical network consisting of the main steps or components in the chosen proof strategy. The elements of the realisation of the components are called the details of the proof.

In a situation where a student has to construct a proof by herself, she has to decide on a proof strategy, construct the proof in a number of sub-steps and finally provide the details of those steps. When the proof is already made as the case is in a textbook the student has to identify the proof strategy used, the components the proof are made up of and the details of these components. In the proposed definition, the structure of a proof equals the hierarchy composed of the strategy choice, the components and the details. The main steps in a proof are often related in some way, but the details of one component may, besides having a relation to other details in the same component, also relate to details of other components in the structure. Relations between components and relations between details of different components give rise to a network within the hierarchy.

There is a dialectical relationship between structure, components and details. It is not possible to comprehend the proof structure if the components are not known and to identify something as a component implies that it is a component of a larger system. Similar considerations apply to the details of the structure.

Although this construction bears some resemblance with “the structural method” proposed by Leron (1983), where a proof is regarded as composed of levels which again consist of modules containing “one major idea of the proof”, it is however different. In the structural method the first level contains the big lines in the proof without any technical details, whereas the last level contains all the specifics in the proof. The components in a proof, as defined in the proposed theoretical construction, can therefore not be equated with the levels in the structural method. And more importantly, the structural method is not a tool for analysing a proof presentation that is based on the “linear method”, as defined in (Leron, 1983).

With the suggested tool it is possible to analyse if the teacher and the students talk about the structure, the components or the details during the presentation and discussion of a proof in the classroom. Within this theoretical framework it is possible to pose a hypothesis for the larger project: *Confusion about what is structure, components and details in the teacher's dissection of a proof can account for students' difficulties solving tasks.* In this paper I first consider a concrete textbook proof with the intention to identify structure, components and details and then I present the analysis of the presentation of the proof in class.

ANALYSIS AND CHARACTERISATION OF A TEXTBOOK PROOF

The analysis course I observed used the textbook “An Introduction to Analysis” (Wade, 2004), so I stick to the formulation of the proof from this book. The exact wording of the theorem is:

> **Theorem 3.6** [Sequential Characterisation of Limits]
>
> Let $a \in \mathbf{R}$, let I be an open interval that contains a, and let f be a real function defined everywhere on I except possibly at a. Then
>
> $$L = \lim_{x \to a} f(x)$$
>
> exists if and only if $f(x_n) \to L$ as $n \to \infty$ for every sequence $x_n \in I \setminus \{a\}$ that converges to a as $n \to \infty$. (Wade, 2004, p. 60)

The proof of theorem 3.6 is given below. The proof refers twice to an implication (1) from the definition of limits of functions, definition 3.1:

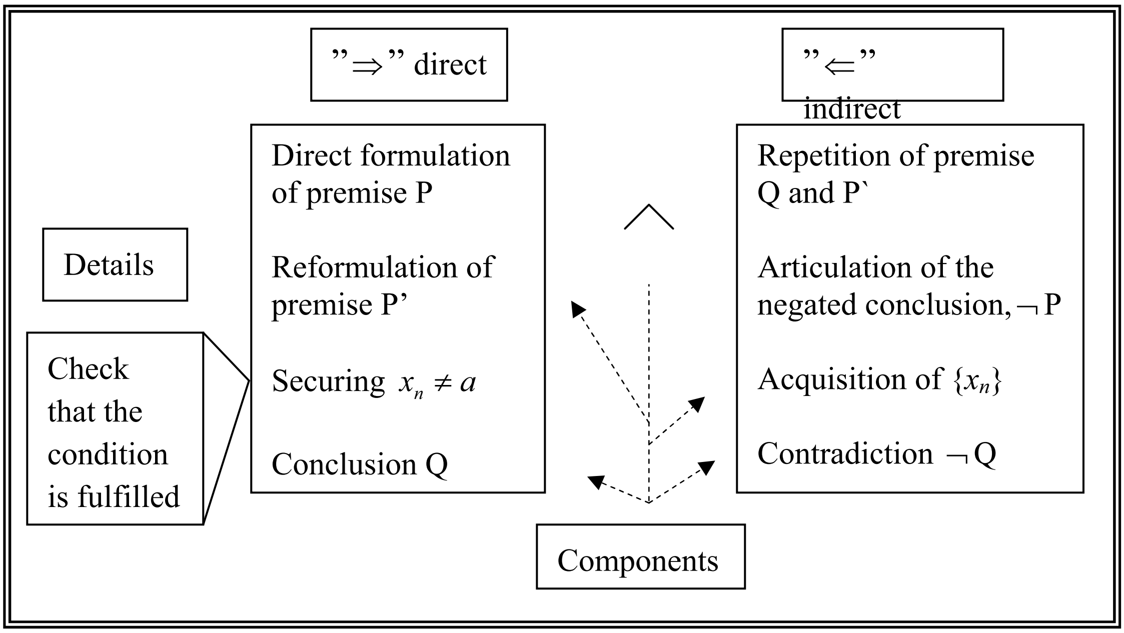
> **Definition 3.1**
>
> Let $a \in \mathbf{R}$, let I be an open interval that contains a, and let f be a real function defined everywhere on I except possibly at a. Then $f(x)$ is said to *converge to L*, as *x approaches a*, if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ (which in general depends on ε, f, I and a) such that
>
> (1) $0 < | x - a | < \delta$ implies $| f(x) - L | < \varepsilon$.
>
> (Wade, 2004, p. 58)

I have included numbers in the proof as a help for the analysis, but besides those numbers the proof is a verbatim reproduction of the textbook proof:

> **Proof**
>
> **1)** Suppose that f converges to L as x approaches a. Then given $\varepsilon > 0$ there is a $\delta > 0$ such that (1) holds. **2)** If $x_n \in I \setminus \{a\}$ converges to a as $n \to \infty$, then choose an $N \in \mathrm{N}$ such that $n > N$ implies $| x_n - a | < \delta$. **3)** Since $x_n \neq a$, **4)** it follows from (1) that $|f(x_n) - L| < \varepsilon$ for all $n > N$. Therefore, $f(x_n) \to L$ as $n \to \infty$.
>
> **5)** Conversely, suppose that $f(x_n) \to L$ as $n \to \infty$ for every sequence $x_n \in I \setminus \{a\}$ that converges to a. **6)** If f does not converge to L as x approaches a, then there is an $\varepsilon > 0$ (call it ε_0) such that the implication "$0 < |x-a| < \delta$ implies $| f(x) - L | < \varepsilon_0$" does not hold for any $\delta > 0$. **7)** Thus, for each $\delta = 1/n$, $n \in \mathrm{N}$ there is a point $x_n \in I$ that satisfies two conditions: $0 < | x_n - a| < 1/n$ and $| f(x_n) - L | \geq \varepsilon_0$. **8)** Now the first condition and the Squeeze Theorem (Theorem 2.9) imply that $x_n \neq a$ and $x_n \to a$, so by hypothesis, $f(x_n) \to L$, as $n \to \infty$. In particular, $| f(x_n) - L | < \varepsilon_0$ for n large, which contradicts the second condition. (Wade, 2004, p. 60)

Theorem 3.6 includes a bi-implication ("if and only if") and the majority of proofs of such theorems are structured in two parts where one implication is showed at a time. The chosen strategy is to prove the first implication "$\Rightarrow$" with a direct proof whereas the second implication "$\Leftarrow$" is proved indirectly by contradiction. To make a strategy choice or to understand why a given strategy choice has been made is an important part of a strategy discussion. In the textbook this strategy choice is not emphasised or discussed.

The theorem has a twist because there is a double hypothesis part. There are thus *two* premises in the first part; P: "$f(x) \to L$ *as* $x \to a$" and P`: "$x_n \to a$ *as* $n \to \infty$" and a conclusion Q: "$f(x_n) \to L$ *for* $n \to \infty$". The proof strategy of the first implication is thus: "if P and P`, then Q", i.e. (P$\wedge$P`)$\Rightarrow$ Q. In the first step the premise P is directly formulated, while premise P` is reformulated in the second step. One might at first glance think that the two steps are similar, but the second step deviates from a mere formulation of the premise. It draws the consequences of premise P` and is in that sense a reformulation of P`. The third step provides the missing link before the results so far can be combined; namely securing that $x_n \neq a$. In the fourth step, the combination of the formulation of premise P, the reformulation of premise P` and the securing leads to the conclusion that $f(x_n)$ converges to L. The structure of the proof with the described components is shown in figure 1.



Figur 1. The structure of the proof is composed of the main steps or components that the chosen proof strategy leads to. The realisations of the components are the details of the proof. The details are shown for one of the components as an illustration. The notation used is: P: $f(x) \to L$ as $x \to a$; P`: $x_n \to a$ as $n \to \infty$ and Q: $f(x_n) \to L$ as $n \to \infty$.

In the second part of the proof an indirect proof strategy, proof by contradiction, is chosen for non-explicit reasons. P` is still a premise, but now Q is a premise and P is the conclusion. The logical structure of this part is based on the logical tautology [(Q∧P`∧¬P)⇒¬Q] ⇒ (Q⇒P). Since the data excerpts only concern the first part of the proof I will not go further into the analysis of the second part.

What does it take to realise the different components? What are the details? I will give some examples. In the first component the formulation of premise P demands a reproduction of the definition of the limit of a function, which includes a repetition of the definition and a switch between the different formulations, phrases and notations used to describe limits of converging functions. The details of the component where we make sure that $x_n \neq a$ (the third component) is just a contemplation that this condition is fulfilled.

It is not uniquely determined what should be the content of the structure, components and details in a proof. We shall see, however, that the proposed characterisation can help characterise what is going on in the classroom.

ANALYSIS AND CHARACTERISATION OF THE CLASSROOM PRESENTATION

Data for the project was constructed through non-participant observations (Bryman, 2001) of a four month long traditionally taught real analysis course at a Danish university. I have selected two sequences from the 25 minutes long presentation of the above proof. Between the excerpts I give a summary of what takes place in the classroom in the non-documented periods so the reader will get a sense of the whole proof presentation. The students were expected to have read or browsed through the proof before the lecture and they were not going to have a test in the proof.

The teacher begins the proof with a claim that proving the first implication is almost trivial (30-33). He says that since they have to talk about all sequences they need to pick an arbitrary converging sequence and see what they can say about that one (34-38). Then he proceeds to make a graphical illustration of the situation (39-40). We enter the scene where he comments on his illustration (the teacher uses a different notation than the textbook, a instead of L and x_0 instead of a). In the excerpt the teacher hastily goes through the first two components (41-42). Then he jumps to the component of the conclusion Q (43) and finally back to the details of the second component (44-46):

41	Teacher:	We have a graph f. We have an ε window. We have a δ which matches. … and we have a sequence, eh, x_n converging down to x_0 and we want to show that the function values of the sequence converge to a, right? And what does it mean that the sequence converges to x_0? … well, then it has to stick to this interval, minus δ to δ, as long as n is big enough. Mary, isn't it?
47	Mary:	I was just gone there for a moment ..
48	Teacher:	You were just, yes, okay. We want to show that the sequence of function values f of x_n converges to a and what we know is that if x is in the δ interval around x_0, then all the function values are in the ε interval around a. And then I say, if we are to make sure that $f(x_n)$ is at most ε away from a then it is basically enough to capture x_n in this interval from minus δ to δ because then we know that the function values are in the right interval … and there .. Can we make sure that x_n is in the interval from $x_0-\delta$ to $x_0+\delta$?
56	Susan	Has it something to do with choosing an n that is big enough?
57	Teacher	That sounds like a really good idea. Can we do that?
58	Susan	We can do that.
59	Teacher	We can do that. What, eh, how big does it have to be?
60	Tom	Bigger than capital N.
61	Susan	Yes, it has to be bigger than capital N.
62	Teacher	No, it's capital N that we are about to choose, right? How big are we going to choose capital N?
64	Paul	So big, that the difference between the sequence and the limit is less than, numerical, less than δ.

66	Teacher	Than δ …

After Mary's sign of lack of attention (47) what is the teacher then doing? He begins "backwards", starting with conclusion Q (48-49) which is followed by the first component, "formulation of premise P" (49-51). Then he tacitly reformulates the logical structure of the proof (51-53): "*if* Q needs to be true, *then* it is enough if P` is true". Instead of talking about the necessary condition for Q to be true ("if P and P`, then Q"), he now focuses on a sufficient condition and that draws attention to premise P` instead of conclusion Q. It is (presumably) very difficult for a student to follow this equivalent reformulation when the teacher does not explicate what he is doing.

The teacher involves the students on five occasions (in 45-46, 54-55, 57, 59 and 62-63). On two of those occasions (54-55 and 57) he poses a question where a proper answer would refer to the second component, "reformulation of premise P`": "*yes, because* $\{x_n\}$ *is chosen to be a converging sequence*". The first reply from Susan refers in stead to the details of this component and in her second reply she does not justify her answer. On the three other occasions the teacher asks with reference to the details of the second component and this is also the response he gets from the students.

This way of analysing the excerpt shows that the teacher aside from tacitly reformulating the logical structure of the proof also shifts between a component perspective and a detail perspective. The students maintain a focus on the details.

The teacher writes down the details of the first two steps (66-72). The following excerpt concerns the securing component. The details of this step only include an inspection which explains why the teacher characterises this step as "free" (75):

73	Teacher	… And then I quickly just want to add, that zero is less than the distance from x_n to x_0 and that is because my sequence will never reach the value x_0, right? That is just for free.
76	Susan	That is just for free?
77	Teacher	Yes, that is, it's just there, my sequence was contained in I without x_0, so none of the x_n's can be x_0.
79	Susan	Why is that free?
80	Teacher	Well, I mean, that assures me that the distance is bigger than zero. That's what's free. When I have paid the other price first, right?

Supposedly, Susan does not realise the details of this component because the structure of the proof is not clear to her and she does not recognise what role the component plays in the structure. Her uncertainty about the structure and to which part of the structure the discussion is located makes it impossible for her to comprehend the details of this component.

The teacher finishes the first part of the proof (82-86) and they have a discussion about the notation (87-114). The teacher moves on to the second part of the proof where he proclaims that he wants to make it as a proof by contradiction if none of the students have any other suggestions (126-128). So in his presentation of the proof no emphasis is put on the justification of the strategy choice in neither the first nor the second part of the proof. After repeating premise Q and conclusion P (129-132) the teacher guides the students through the details of the component “articulation of the negated conclusion” (133-165). Here both the teacher and the participating students are talking about and referring to the details and it is clear from the transcripts (not shown) that the students are able to follow his guiding. A reason for this accordance may be that the students recognise the link between the strategy choice and the negation component and thus are able to understand the explanation of the details.

After guiding the students through the details of the negation component the teacher continues to the seventh step, the “acquisition component”, which leads to difficulties for the students. He begins with a repetition of premise P` and Q (166-167). A student expresses difficulties with the choice of the sequence *1/n* and the teacher tries to explain it while maintaining a focus on the details (184-209). After trying to explain the acquisition component the teacher interprets a question from a student as a formulation of the contradiction component (210-215) and the teacher quickly summarises the components of the second part of the proof (215-220) and writes down a formal version of his summary focusing on the details (216-237). Then Susan expresses some confusion about the structure of the proof; didn’t they assume what they were trying to prove? This leads to a clarification of the logical structure of the second part of the proof (238-247) and of the logical structure of a proof of an arbitrary “if-then” theorem (248-254).

SUMMARY

The two chosen excerpts show episodes where the teacher and the students in some way miscommunicate, but I briefly mentioned one example where the students and the teacher had a united approach, namely in the formulation of the negated conclusion. In the first excerpt the teacher jumps around between the components and he reformulates the logical structure of the first part of the proof. To a student who does neither comprehend the structure of the proof nor is able to separate the components from each other it must be nearly impossible to follow the presentation and to comprehend the details. In the second excerpt the teacher explained the details with reference to the (underlying) structure. In order to understand why the fulfilment of the relation is “free” it is necessary for the student to see what role this component plays in the proof. In the last part of the presentation two students expressed difficulties understanding the details of the seventh component even though the teacher tried to explain the details. Utterances in the excerpt and later in the presentation indicated that the proof structure was not clear to at least one of the students, so again it is possible to conclude that a lack of understanding of the proof structure and the components prevents a comprehension of the details.

As mentioned, the notions in the framework are dialectical related. As the analysis of the transcripts shows, this dialectical relationship is in fact visible in the students’ struggles to comprehend the proof.

Research studies show that university students typically exhibit difficulties handling quantification (Dubinsky, 2000). These difficulties are not directly addressed in this framework. The framework has been constructed through a "bottom-up" process founded on data and it is thus context dependent. Difficulties with quantification did not appear to be as essential in explaining the students' difficulties as their struggle to separate between structure and details in the dissection of a proof in class.

NOTES

1. By teaching practice I am referring to activities taking place in the course session time, to the organisation of the course, to the choice of textbook, to the way the subject matter is presented and to communication in class.

2. The framework developed by Cobb and co-workers takes both the social and the individual perspective into account (Cobb et. al, 1997). I did not find this framework completely useful for my data analysis because of a lack of focus on solving processes. I however found use of this framework for parts of the data analysis (not reported here).

ACKNOWLEDGEMENTS

I am very grateful to Mogens Niss for his valuable comments, ideas and suggestions during the preparation of this paper.

REFERENCES

Alibert D. and Thomas, M. (1991) Research on mathematical proof. In D. Tall (ed.) *Advanced Mathematical Thinking* (pp. 215-230), Kluwer Academic Publishers.

Balacheff, N. (1991) The benefits and limits of social interaction: The case of mathematical proof. In A. Bishop et.al. (eds.) *Mathematical Knowledge: Its Growth Through Teaching* (pp. 175-192), Kluwer Academic Publishers.

Bourbaki, N. (1950) The architecture of mathematics. *The American Mathematical Monthly* **57** (4), 221-232.

Bryman, A. (2001) Social research methods. Oxford University Press. USA.

Cobb, P., Gravemeijer, K., Yackel E., McClain, K. and Whitenack, J (1997) Mathematizing and symbolizing: The emergence of chains of signification in one first-grade classroom. In D. Kirshner et.al. (eds.) *Situated Cognition. Social, Semiotic and Psychological Perspectives* (pp. 151-235), Lawrence Erlbaum.

Dreyfus, T. (1991) Advanced mathematical thinking processes. In D. Tall (ed.) *Advanced Mathematical Thinking* (pp. 25-41), Kluwer Academic Publishers.

Dreyfus, T. (1999) Why Johnny can't prove. *Educational Studies in Mathematics* **38** (1-3), 85-109.

Dubinsky, E. and Yiparaki, O. (2000) On student understanding of AE and EA quantification. In E. Dubinsky et.al. (eds.) *Research in Collegiate Mathematics Education IV* (pp. 239-286). AMS and MAA.

Hanna, G. (2000) Proof, Explanation and exploration: An overview. *Educational Studies in Mathematics* **44** (1-3), 5-23.

Harel G. and Sowder, L. (1998) Students' proof schemes: Results from exploratory studies. In A. Schoenfeldt et.al (eds.) *Research in Collegiate Mathematics Education 111* (pp. 234-283), MAA.

Healy, L. and Hoyles, C. (2000) A study of proof conceptions in algebra. *Journal for Research in Mathematics Education* **31** (4), 396-428.

Hemmi, K. (2006) Approaching proof in a community of mathematical practice. Ph.D. Dissertation. Stockholm University. Available online (date 4th January 2007) at: http://www.diva-portal.org/diva/getDocument?urn_nbn_se_su_diva-1217-2__fulltext.pdf).

Legrand, M. (2001). Scientific debate in mathematics courses. In D. Holton (ed.) *The Teaching and Learning of Mathematics at University Level* (pp. 127-137), Kluwer Academic Publishers.

Leron, U. (1983) Stucturing mathematical proofs. *The American Mathematical Monthly* **90** (3), 174-185.

Martin, W. G. and Harel, G. (1989) Proof frames of pre-service elementary teachers. *Journal for Research in Mathematics Education* **20** (1), (41-51).

Moore, R. C. (1994) Making the transition to formal proof. *Educational Studies in Mathematics* **27** (3), 249-266.

Raman, M. (2003) Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics* **52** (3), 319-325.

Selden A. and Selden, J. (2003) Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education* **34** (1), 4-36.

Wade, W. R. (2004) *An introduction to analysis*. Pearson Prentice Hall, USA. Third Edition.

Weber, K. (2001) Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics* **48**, 101-119.

Weber, K. (2004) Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductionary real analysis course. *Journal of Mathematical Behavior* **23**, 115-133.