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TOOLS AND TECHNOLOGIES IN MATHEMATICAL DIDACTICS

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Introduction

This working group focuses on a theme which has been attracting the interest of researchers in mathematics education for many years, long before the first CERME conference. In the eighties, there was an early focus on questioning mathematics education as a whole by means of illuminating studies used to articulate a visionary picture of schooling based on doing mathematics with technology. This was quickly challenged by grounding comparative studies in turn used to doubt the potential of quick radical changes in schools and in students' mathematical abilities. In the nineties, epistemological analyses of the mathematics embedded in a range of software tools emerging from the rapid developments in technology, as well as domain – specific studies of these tools in use gave a picture of a more complex world through case studies of changing classrooms. Research in learning with technological tools has indeed been showing much promise with respect to their use in the generation of learning environments where students have richer opportunities to construct mathematical meanings, to explore and experiment with mathematical ideas and to express these using a wealth of representations. However, actual use of these tools in real school environments is still very thin despite the abundance of governmental funding and interest at the European level. The changes in classroom practices involved in the use of technology seem to pose a real challenge to administrators, curriculum designers, teachers and students. There is thus a great need for a deeper understanding of how the potential suggested by research in the 80s and the 90s can be grounded both in classroom practices with respect to systemic schooling, in other institutional environments such as the workplace and in informal situations for children and adults. The use of technologies has simply not scaled up and the changes promised by the case study experiences have not really been noticed beyond the empirical evidence given by the studies themselves. In recent years, focus has also turned on teaching methods and on ways of supporting teachers to use technological tools.

Even from the previous CERME, discussing about the integration of technological tools in mathematic classrooms appears as inevitable: “The integration of tools and technologies is an important and actual theme today”, as pointed out by Paul Drijvers and his team (Proceedings of CERME 4, p. 927). This has resulted in a much more careful reference to changing schooling and the development of a more refined language to address the use of technologies in mathematics education (Ruthven, Artigue, this volume). In an attempt to contribute to the development of this language, the working groups in CERME 4 and 5 were organized in themes. In WG 4 there were three such themes, the tool, the teacher and the learner. In WG 5 we organized

the papers and our work into four themes portraying important aspects of the work of the group. These themes portrayed the primary concern in the papers written. They were placed on two axes, one was the teacher – student axis and the other the artifact – instrument axis, i.e. focusing on the tool itself or on the use of the tool. It goes without saying that the axes cannot be used in a positivistic, Cartesian way. However, they did help the group elaborate a noticeably succinct language around digital technologies and their use in mathematics classrooms. The Working Group was concerned with the use of technology in mathematics education at all levels of systemic education, ranging from primary school up to university teaching. It is also concerned with the use of technology for learning mathematics in the workplace and in informal settings.

In the final session of the working group, we used the axes as a geography on which to place research questions emerging from pairs of researchers not familiar with each other's work (see figure 1). This was done as an exercise (an enjoyable one) to be more explicit about researchers' concerns regarding those four themes. The group also engaged in discussing the placing of the papers on the system which resulted in a number of issues coming to the foreground of our attention (figure 2).

Finally, in the WG of CERME 5 we introduced two more activities. One involved the presentation of European research projects in the field and the other the demonstration of artifacts recently developed or under development and a discussion of their affordances. It was felt that, apart from informative, these sessions were very useful in generating the feeling of being part of the European scene in the field. Presenting and discussing research in progress and artifacts under development seemed fruitful for the theme of this group where developments are rapid and a lot is going on.

- How does the interplay between students' physical movement and direct manipulation of graphs build meaning of the concept of function? *Psycharis - Jones*
- How does teachers preferred choice of artifact according to their preferred problem solving approach influence students' problem solving approach? *Baerzel – Zsolt*
- How does ICT affect 'traditional' student mathematical skills? *Bardini – Drijvers*
- Is it appropriate for teachers to subvert the use of artifact and if so, when? *Averis, Lozano*
- How to construct a technological environment for the development of probabilistic thinking? *Weigand – Sivirianou*
- How can children's concept of mathematics be expressed via ICT? *Melusova – Fuglestad – Vidermanova*
- How do prospective teachers view the image of the good teacher while interacting with the geometry supposer? *Sever - Lavy*
- In phase of the educational process (motivation – learning...) should the teacher decide to use the computer in the lesson? *Kraslanova Slavickova*

Figure 1 - Some questions addressed by the members of the group

	teacher Sanne - Guedet	Buenoravel - Assude
		Averis, Lozano
		Sever - Lavy
artifact	Kraslanova Slavickova	instrument
	Hegedus - Picard	Maffei - Emprin
		Baerzel – Zsolt
	Weigand – Sivirianou	
	Bardini - Drijvers	Psycharis - Jones
		Latsi - Biza
		Melusova – Fuglestad - Vidermanova
	learner	Alexopoulou - Christou

Figure 2 – Positioning the questions in the two-dimensional structure

This two dimensional structure also helped us map the different papers as shown in Figure 3:

	teacher		Averis Emprin
			Bueno-Ravel
			Assude
Dana-Picard		Biza	instrument
artifact			
Oldenburg		Fuglestad	
Christou et al			Drijvers et al
			Hegedus et al
	Slavickova	Lozano et al	
		Lavy	Barzel
	Weigand		Psycharis et al
	Misfeld et al	Pitta et al	Kynigos et al
Maffei		Sivirianou	Sever et al
	learner		

Figure 3 – Positioning the papers in the two-dimensional structure

From this two-dimensional structure, four sub-groups naturally emerged and shaped the organization of the workshop: each member of the group either joined the “Teacher”, the “Learning”, the “Artifact” or the “Instrumentation” sub-group.

Some of the questions raised by “Teacher” sub-group concerned:

- Instrumental genesis for teachers (how does this theory adapt to the teachers case?),
- Pre-service teacher education (how does pre service teachers change their view of teachers in ICT settings?)
- In-service teacher education (what kind of orchestration in teachers training?)
- Affordance awareness (how can interactive tools such whiteboards help? How to define a “degree of integration”?)

The subgroup was particularly concerned with the different roles teachers adopt when using technology (teacher as mediator, instructor, etc.), some particular roles assigned to ICT (e.g. ICT for communicating), the changes in students’ maths abilities, the evolution in teacher practices (scenarii seem to be more individualized, more constrained) and also the change in prospective teachers’ view engaged in computerized environment regarding the image of the teacher itself.

The “Instrumentation” and “Artifact” sub-groups were initially driven by open questions such as the instrumental genesis of artifacts : What should an artefact look like for an easy instrumental genesis for students ? (Multiple representations, Interactivity...) What should an artifact look like for an easy instrumental genesis for teachers? (Teachers may prefer “small specifics” when beginning to include technology in teaching, the broader influence by using “generals”).

Some more specific issues were also addressed: the relationship between individual cognitive styles and performance in geometrical tasks, the role of metaphors and dynamic representations in dealing with students’ active construction of meanings and the role of *design of artifacts* providing tools that can be viewed as *integral to mathematical activity (artifacts-in-use)* rather than an external aid to internal cognitive processes (e.g. what kinaesthetic and conceptual aspects are important to take into account when developing softwares?).

By the end of the workshop, the groups seemed to agree to conceive instrumentation both as a lense to frame the theoretical frameworks and as a framework itself. Also, some affordances of artifacts that enhance instrumentation were mentioned, namely: to constitute exploration spaces, mediate between informal and formal, provide executable representations, offers dynamic manipulation, evoke interplay between private and public expression and to generate interdependent representations.

On the side of the “learners”, long-term studies were reported, and questions such as maths’ abilities with symbolic calculator were raised, the gender difference on students’ performances, and the integration of in the curriculum across Europe.

This introduction concludes with a slide for each one of the research projects and the artefacts presented in the WG.

CERME 5 WG 9 Larnaca 21-26 February 2007

ICT and mathematics learning (ICTML)

- Developing inquiry communities in mathematics
- Key activities:
 - Workshops with teachers and didacticians
 - School team meetings discussing teaching approaches and lesson plans. Work in classes (grades 8 - 10)
 - Observing, reflecting, giving feedback
- Developmental research:
 - design cycle/inquiry cycle
 - research on all levels



Anne Berit Fuglestad
Agder University College,
Kristiansand, Norway <http://faq.hia.no/iktml>

CERME 5 WG 9 Larnaca 21-26 February 2007



ReMath FP6-IST

To contribute to the integration of theoretical frames for learning with digital media by generating a meta-language based on experience of being explicit about the process of cyclical development of six dynamic digital artifacts

Presenter: C. Kynigos
<http://remath.cti.gr>
RACTI, DIDIREM, IMAG-UJF, LKL, ITD,
ETL/NKUA, Sienna, Talent S.A.

CERME 5 WG 9 Larnaca 21-26 February 2007



Learning Patterns for the Design and Deployment of Mathematical Games

To map, through a set of customisable patterns, the relationship between design and deployment of mathematical games, with the longer term aim of promoting the reuse of proven techniques

Presenter: Efe Alexopoulos
London Knowledge Lab, UK
University of Warwick, UK
ETL, University of Athens, Greece
Freudenthal Institute, The Netherlands
Trinity College Dublin, Ireland
Göteborg University, Sweden
Istituto per le Tecnologie Didattiche, Italy
<http://www.noe-kaleidoscops.org>

CERME 5 WG 9 Larnaca 21-26 February 2007

GUPTEN: Genesis of Professional Uses of Technologies by Teachers

- Presenter: G. Gueudet
- A french national project, directed by Jean-Baptiste Lagrange.
- Significant uses of ICT require an important evolution of the teachers' practices and beliefs: geneses.
- Geneses have an external aspect (evolution of practices) and an internal aspect (evolution of beliefs).
- Geneses' features can be analysed along three axes: skill/use; change/resistance; institutional incitation/appropriation.

CERME 5 WG 9 Larnaca 21-26 February 2007



Teaching Calculus using Dynamic Geometric Tools SOCRATES - Comenius Action 2.1 Training of School Educational Staff

Presenter: E. Biza

- The objectives of this project are:
- the production of an instructional proposal for the functions and Calculus in secondary mathematics education supported by dynamic geometrical tools
 - the design of an in-service training program for teachers based on the above proposal.



PARTNERS

- University of Athens, Greece (coordinating institution)
- University of Crete, Greece
- University of Southampton, U.K.
- University of Sofia, Bulgaria
- University of Cyprus

<http://www.math.uoa.gr/calgeo>

CERME 5 WG 9 Lamaca 21-26 February 2007

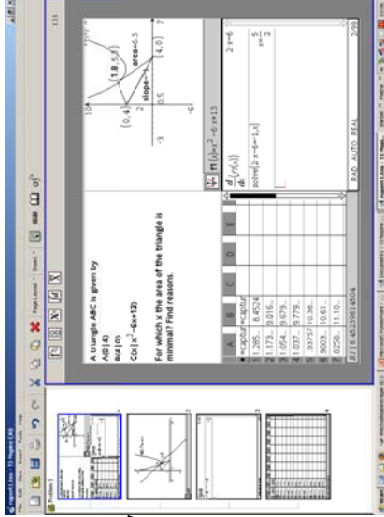
TI-Nspire CAS

(published: mid2007)

An interactive combination of :

- Dynamic Geometry
- Computer algebra
- Spreadsheet

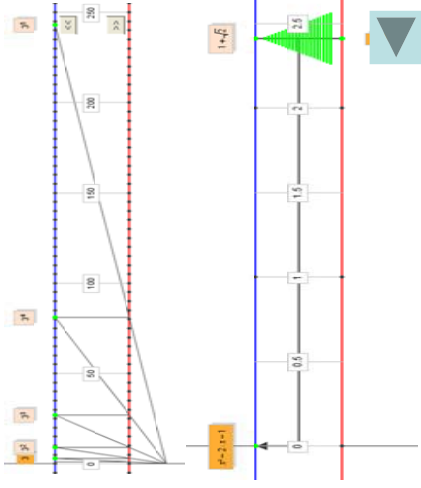
Available on Handheld & PC (Linked by USB-stick)



CERME 5 WG 9 Lamaca 21-26 February 2007

Alnuset

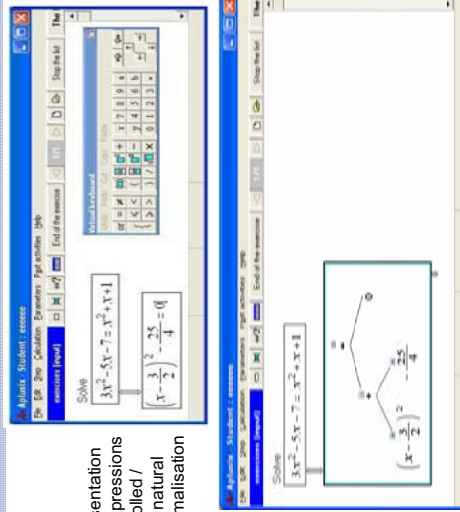
- algebraic line component:
- representation of numerical sets (natural numbers, relative numbers, rational numbers, irrational numbers of the form $\frac{ap}{q}$):
- algebraic manipulator component



CERME 5 WG 9 Lamaca 21-26 February 2007

Aplusix

- tree representation of algebraic expressions
- Free /controlled / mixed tree – natural algebraic formalisation



CERME 5 WG 9 Lamaca 21-26 February 2007

Cruislet

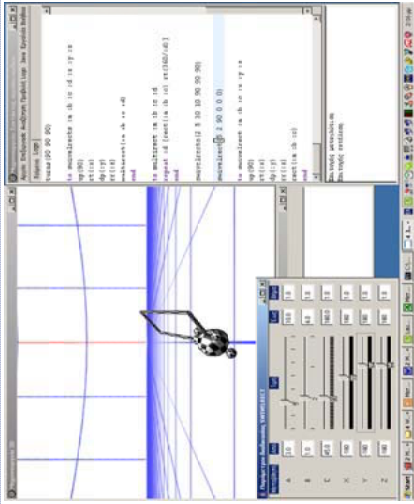
- vectors as displacements in geographical and polar coordinate systems
- navigating objects in 3d maps with geo-coded information



CERME 5 WG 9 Larnaca 21-26 February 2007

MachineLab

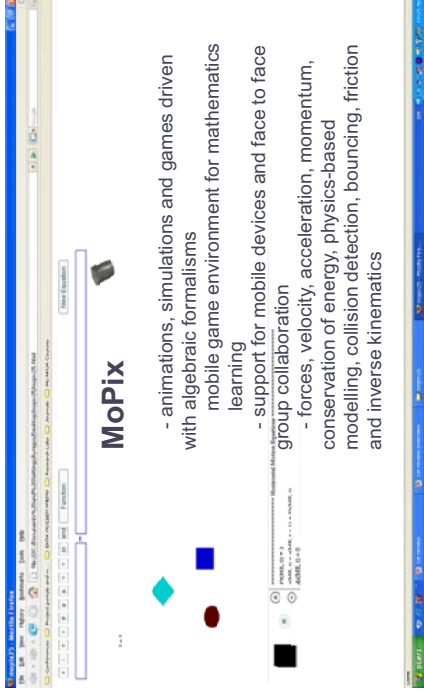
- a vectorial approach to a 3D Turtle Geometry environment
- integration of programmability with dynamic manipulation of variable procedure values and resulting geometrical figures
- Built on a 3D game engine and a Java-based Logo compiler



CERME 5 WG 9 Larnaca 21-26 February 2007

MoPix

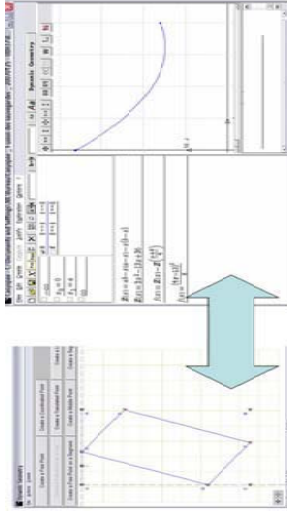
- animations, simulations and games driven with algebraic formalisms
- mobile game environment for mathematics learning
- support for mobile devices and face to face group collaboration
- forces, velocity, acceleration, momentum, conservation of energy, physics-based modelling, collision detection, bouncing, friction and inverse kinematics



CERME 5 WG 9 Larnaca 21-26 February 2007

Casyopee

- link enactive and theoretical representations
- algebraic manipulations on functions;
- modelling dynamic geometrical dependencies with functions



Teacher's practices and degree of ICT integration

Teresa Assude

UMR ADEF, Université de Provence

In the presentation of the working group 9 “Tools and technologies in mathematical didactics”, we can see how deep is the gap between the research results on the use of technology in the mathematical learning (“richer opportunities to construct mathematical meanings”, “explore and experiment with mathematical ideas”) and the little use of these technologies in the real classroom (“actual use of these tools in real school environments is still very thin despite the abundance of governmental funding and interest at the European level”). This gap is problematic and “pose a real challenge to administrators, curriculum designers, teachers and students.”

To take up this challenge, teachers have been coping with professional problems facing institutional injunctions for integrating ICT in classrooms. How do they implement these injunctions? What do they propose for pupils' activities? There are different ways of working out this professional problem. Some teachers simply put it apart and do nothing, others bring some minimal strategies into play, and others invest very much to achieve this integration. How can we distinguish these different practices? How do we characterize them? I have tried to answer these questions looking at the integration of a dynamic geometry software (Cabri-géomètre) in primary teaching.

My contribution in this study takes place within topic 2 “The role of the teacher in technology-rich mathematics education”. My questions are connected with the question “how teachers can deal with the new pedagogical context of technology-rich learning?”.

While approaching these questions, I carry on two purposes. First, I want to develop a theoretical tool to clarify how the teachers in their practice are “being aware of the constraints and affordances of the available technology”. For this I define the “degree of ICT integration” using indicators such as “mode of instrumental integration” and “mode of praxeological integration” (I will explain these words in section 2). My second purpose is to describe how teachers without experience in using technology work out this complex problem and to define their needs about resources and training (“what pedagogical resources are available or should be developed?”).

In this paper, I will essentially deal with my first purpose, i.e. a presentation of my theoretical tool. I will begin by succinctly presenting the context of the research work (without showing all the aspects of this research such as the role of Cabri in the teachers' professional development or the differences between several teachers' practices). Then I will make clear the theoretical tool and I will end by analyzing an example. This example is just to illustrate the possibilities of the theoretical tool for evaluating and characterizing teachers' practices in the use of technology.

1 – Context of the research work

The work was carried out in the frame of a national project (Technological Research Team in Education) entitled MAGI (“Mieux Apprendre la Géométrie avec l’Informatique”, in English “Better Learning of Geometry with Computers”). The project is a development and research project involving twenty researchers, teacher educators and teachers divided into groups located in different places in France. The aim is to study the process of integration of dynamic geometry software, namely Cabri-geometry, into the ordinary teaching contexts of the primary and beginning secondary school. The project has two parts:

- design and implementation scenarios for the use of Cabri in several classes of primary and secondary schools (in about ten classes);
- study of the impact of teacher preparation sessions for the use of Cabri in their classroom practice.

Dynamic geometry is conceived in this project as a tool for helping students to move from a purely visual conception to the construction of geometrical theoretical concepts such as collinearity, perpendicular, parallel, congruence... That is why the primary and the beginning of secondary school, were chosen for study. In France, the entry to « theoretical geometry » begins in primary school and is achieved in the first years of secondary school. In addition, geometry is not a favoured subject for primary school teachers who view the teaching of geometry as essentially the teaching of a vocabulary and not as the construction of a coherent model of spatial phenomena and objects. Dynamic geometry can change their view of geometry and motivate them to change their teaching of geometry. This is why it seemed particularly interesting to investigate the integration of dynamic geometry at these school levels.

In this context, the subgroup in Toulon worked with primary school teachers who had no knowledge of dynamic geometry before the project. They were introduced to dynamic geometry in a short training session (half a day). We worked with three teachers (Ingrid, Françoise, Robert), in charge of 10 year-old pupils. The agreement was made with the teachers that they would integrate the use of the software in relation to the whole work of their class, that they were completely free to choose activities with the software and that the role of researchers was restricted to observing teachers without intervening in the choice and the design of the activities nor the management of the class. We wanted to bring out the conditions and constraints of integration of the software by ordinary teachers who had to construct « everything » including a relationship with Cabri. Analyses are done by means of observation notes, videos, students’ notebooks, and interviews with teachers. The teachers in Toulon were observed over one year.

The number of sessions devoted to the use of Cabri varied according to the teachers. Françoise proposed 5 sessions : two for introducing students to Cabri and three sessions of reflection. Ingrid proposed 4 sessions : two for introducing students to Cabri (the same ones as Françoise) and two sessions, one on square and one on

triangles. Robert proposed a weekly session from December onwards (altogether about 15 sessions). These sessions dealt with triangles, squares, circles.

2 – Theoretical framework: mode and degree of integration

My analysis of technology integration into teaching is based on a multidimensional approach (Artigue & Lagrange 1998, Artigue 2001, Guin & Trouche 2004, Lagrange 2001, Trouche 2005) that takes into account several dimensions: epistemological, cognitive as well as instrumental, institutional and anthropological. This multidimensional approach shows the complexity of the integration and anticipates the actors' difficulties in using technology: we take into account the instrumental genesis (Vérillon & Rabardel 1995), the institutional constraints, the epistemological changes in mathematical learning and teaching, the interplay between technical and conceptual work. The study made by Lagrange et alii (2003) has shown that there has not been much research about this problematic of integration. The process of integration is not easy because there are too many variables to manage in the classroom: one of these variables is the dialectic between old and new practices (Assude & Gélis 2002). In our case, the work with Cabri implies to know some principles: difference between drawing and figure (Fischbein 1983, Laborde & Capponi 1994, Laborde 1998), the role of dragging, etc.

My work follows all those studies but I use these different dimensions to define some indicators and the theoretical tool “degree of integration”. What do I mean by “degree of integration”? This measures the organization by the teacher of the instrumental dimension and the mathematical dimension, and their relations. However it is difficult in the moment to assign a very precise degree to a teacher's practice and I prefer to define modes of integration, characterizing a teacher practice over a period of time, a session or part of a session, or a sequence of teaching sessions. These modes are:

- the mode of instrumental integration pointing to how instrumental integration is taken into account. I am interested in the teacher's “orchestration” (Trouche 2005), and I'm particularly looking at the types of tasks (Cabri or mathematics), the instrumental knowledge (IK), the mathematical knowledge (MK), the relations (IK/MK);
- the mode of praxeological integration (Chevallard 1997, 1999) pointing to how the pupil's mathematical work is organized: I am particularly looking at the relationship between paper-pencil tasks and techniques and Cabri tasks and techniques;

These modes are associated with three variables: the dialectic between old and new, the didactical contract and the quantity of work. Furthermore, these modes are not independent (instrumental and mathematical dimensions are overlapping one another). I would like to stress here that, as technology involved in mathematics education embodies mathematics, the technical and the conceptual parts are intrinsically intertwined (Artigue *ibid.*): the use of technology shapes the knowledge

constructed by students (Hoyles et al. 2004). But I distinguish these modes for bringing out the different dimensions of integration.

2.1 - Mode and degree of instrumental integration

Until now I have identified four modes of instrumental integration:

- instrumental initiation
- instrumental exploration
- instrumental reinforcement
- instrumental symbiosis.

To determine these different modes, I use several indicators: types of tasks (mathematics (TAM) or Cabri (TAC)), instrumental knowledge (IK), mathematical knowledge (MK), relations between these two indicators (IK/MK). We distinguish between two cases of instrumental knowledge: either pupils are beginners or novice in using the artifact (low IK or no IK) or they already have knowledge of the artifact but they have not yet a good knowledge of how to handle all the facilities (average IK).

Pupils are beginners:

In the *instrumental initiation*, pupils don't know the software and are initiated into Cabri tasks. The teacher's aim is mainly that the pupils learn how to use the software (pupils must learn some IK). The relation between IK and MK is minimal.

In the *instrumental exploration*, pupils do not know the software and are going to explore it through mathematical tasks. The teacher aims at improving both IK and MK. The relation between IK and MK can vary from the minimum to the maximum according to the mathematical task. This mode can evolve to the instrumental symbiosis.

Pupils are not beginners

In the *instrumental reinforcement*, pupils are already used to the software but they are confronted with instrumental difficulties while dealing with a mathematical task. The teacher will give them instrumental information. The teacher's aim is improving mathematical knowledge. The relation between MK and IK is maximal because IK is required to achieve the mathematical task.

In the *instrumental symbiosis*, pupils have already used the software and they are confronted with mathematical tasks which allow them to improve both their IK and MK because they are connected. The relation between IK and MK is maximal:

each one allows the other to increase and the connection between paper-pencil work and Cabri work is strong.

	Initiation	Exploration	Reinforcement	Symbiosis
TAC	x			
TAM		x	x	x
IK	x	x		x
MK			x	x
IK/MK			IK tool to MK	Max

These different modes of instrumental integration lead to classifying the integration practices from the lowest level (where the only integration taken into account is instrumental initiation) to the highest level (where the different modes are taken into account according to the appropriate moment).

2.2 - Modes and degree of praxeological integration

The pupil's mathematical work can be described by mathematical praxeologies (Chevallard 1999). These mathematical praxeologies are quadruplets made up by tasks, techniques, technologies, theories. Technologies here mean a justification of techniques. Our indicators are now praxeologies associated with two other variables: variable "paper-and-pencil/cabri" and variable "old/new":

- types of tasks cabri (TAC) (old/new)
- types of tasks paper-and-pencil (TAPP)(old/new)
- techniques cabri (TEC)(old/new)
- techniques paper-and-pencil (TEPP)(old/new)
- relations between tasks cabri and tasks paper-and-pencil (TAC/TAPP)
- relations between techniques cabri and techniques paper-and-pencil (TEC/TEPP)
- "weak techniques" (meaning techniques without technological and/or theoretical justifications)(WTE)
- "strong techniques" (meaning techniques with technological and/or theoretical justifications)(STE)

Until now I have identified five modes of praxeological integration. A mode is *void* if there are no TAC and TEC in the pupil's mathematical activities. A mode is *minimal* if there are TAC and TEC, and there are no TAPP and TEPP. A mode is *juxtaposed* if there are TAC, TEC, TAPP, TEPP and no relationships between these tasks and these techniques. A mode is *intertwined* if there are TAC, TEC, TAPP, TEPP and there are some relationships between these tasks and techniques and all techniques are weak. A mode is *maximal* if there are all types of tasks and techniques and strong relationships between these tasks and techniques and if there are STE and new rules of didactic contract taking into account the specificity of dynamical geometry software.

	Void	Minimal	Juxtaposed	Intertwined	Maximal
TAC		x	x	x	x
TAPP			x	x	x
TEC		x	x	x	x
TEPP			x	x	x
TAC/TAPP				some relations	x
TEC/TEPP				some relations	x
WTE				x	
STE					x

These instrumental and praxeological modes associated with other variables like as « old and new dialectic », « rules of didactic contract » and the « number of sessions », are a means to define a degree of ICT integration. At this moment, I define four degree: *zero*, *low*, *medium* and *strong*.

A degree of ICT integration is *low* if the instrumental mode is initiation, if the praxeological mode is minimal or juxtaposed and if there is no dialectic between old and new tasks and techniques and no changes in didactic contract. A degree of ICT integration is *medium* if the instrumental mode is initiation and reinforcement, if the praxeological mode is juxtaposed or intertwined, and if there are some relations between old and new and some changes in didactic contract. A degree of ICT integration is *strong* if all those dimensions are implemented in classroom.

3 – A low level of instrumental integration: an example

In this section, I use the different modes and indicators to analyze an example of teacher's practice. A teacher - Françoise - has proposed two initiation sessions to the pupils. During these sessions pupils learn some IK: creating and moving a point, creating and moving a straight line, creating and moving a circle, creating a segment, naming points. They must read a form indicating all the actions they need to do and there is no collective institutionalization of these IK. The status of points is not identified, and though there are moves, the teacher does not insist on moving to verify the constructions.

In these sessions, the type of task is a Cabri task whose aim is to build and move some mathematical objects. The relation between IK and MK is minimal and so is the connection between Cabri tasks and paper-pencil tasks. We have here an instrumental initiation. Besides, that initiation does not insist on changes in the didactical contract: the function of moving the constructions and the contribution of Cabri to an experimental approach of geometry are not told. That initiation is aimed at building mathematical objects instead of emphasizing the kind of work the software allows with those objects. In the following sessions the teacher is reluctant to use both the instrumental reinforcement (although it sometimes appears) and the instrumental symbiosis.

The mode of instrumental integration is limited to an instrumental initiation and that fact leads us to make the hypothesis that during the first year in which that

teacher tries to integrate Cabri software, she does not pay enough attention to the instrumental dimension (although there are initiation sessions). From this point of view the degree of instrumental integration is low.

4 – A low level of praxeological integration : an example

Françoise proposed two initiation sessions and three sessions about reflection (two sessions in technology classroom and one session in traditional classroom) to the pupils. The first session began by remembering some IK, followed by activities about reflection with the title “Go to play”. An example of these activities follows:

1) Create a point A and create a straight line (MN)
For this, select the tool “point” in the box “creation”
Don’t forget to name it A
Start again and create the points M and N (naming them immediately)
Select the tool “line” in the box “creation”
Approach the cursor to M: press when the message “through this point” appears
Approach the cursor to N: press when the message “through this point” appears
2) Construct the image B of the point A by reflection with respect to the line (MN)
For this, select the tool “reflection” in the box “construction”
Approach the cursor to A: press when the message “reflect this point” appears. The point A is winking.
Approach the cursor to the line (MN): press when the message “with respect to this line” appears.
Name this new point B
Move the point A

Question 1: What’s happened to the point B?
Question 2: Where must you put the point A so that A and B are the same?

The aim of these sessions is the exploration with the software of the notion “reflection”. The pupils are working in pairs and each has a paper with the instructions. The type of tasks are the following:

T1: construct the image of a point by reflection

T2: identify some properties about reflection

For these pupils, T1 is an old task and T2 is a new task. In the paper they have, construction techniques are described in a lot of detail and controlled: all actions are indicated as in the example above. Pupils just need to follow the instructions. To identify some properties, pupils must observe drawings and complete some sentences like this: “*Complete: the image of a segment by reflection is a with the same*”

Pupils used the techniques proposed by the teacher but they did not understand why these techniques were relevant. They had many difficulties in identifying and completing the sentences because they followed the indications step by step, and they

did not have a global activity vision. This strong control prevents pupils from facing some instrumental difficulties and acquiring mathematical knowledge but it is also how the teacher manages the integration of Cabri (she is working for the first time with the software). She is strongly controlling the pupils' work to avoid any questions she could not answer.

There are not any relations between PAC and PATT, and so there are not any relations between PEC and PETT: the work with the software is isolated, all the techniques are WTE, and there is not any reflection about the specificity of dynamic geometry software.

In this classroom, the mode of praxeological integration is minimal and the degree of praxeological integration is low. Françoise wanted to integrate Cabri but she controlled the pupils' work by close activities and "technique algorithms".

5 – Conclusion

My work allowed me to characterize some integration practices from a set of indicators. I used these indicators to analyze three teachers' practices and we presented here an example using Françoise's classroom¹. It can seem limited but it permits me to develop a theoretical tool and to make an hypothesis.

For next works, my hypothesis is the following : a low degree of integration means that the instrumental dimension is not sufficiently taken into account ; there is not a good interaction between the paper-and-pencil activities and the software activities, and no dialectic between old and new. A medium degree of integration means that one of these dimensions is taken into account but not the others. A strong degree of integration means that all those dimensions are implemented in the classroom. The number of sessions is also an important variable – few sessions with software do not allow a good integration - but a justifiable number of sessions is not a sufficient condition for a strong degree of integration if the other indicators are not verified.

In the future, I will use these indicators to evaluate and characterize the practices of teachers well trained in the use of technology. I will compare the degree of ICT integration and I will verify if the low degree is specific of inexperienced teachers. In any case, the work with these teachers has shown me that teachers training and resources must insist on all the dimensions and particularly the instrumental one, and the relationship between instrumental and mathematical dimensions.

References

Artigue M. & Lagrange J.-B. (1999), 'Instrumentation et écologie didactique de calculatrices complexes: éléments d'analyse à partir d'une expérimentation en classe de Première S', in D. Guin (ed) *Actes du congrès "Calculatrices*

¹ Other teachers' practices analyses can be seen in Assude 2006, Assude, Grugeon, Laborde & Soury-Lavergne 2006, Assude & Grugeon 2006.

- symboliques et géométriques dans l'enseignement des mathématiques* ", IREM de Montpellier, pp. 15-38.
- Artigue, M. (2002) Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work *International Journal of Computers for Mathematical Learning* 7(3), 245-274
- Assude T. & Gélis J.M. (2002), Dialectique ancien-nouveau dans l'intégration de Cabri-géomètre à l'école primaire, *Educational Studies in Mathematics*, 50, 259-287.
- Assude T (2006), Degré d'intégration de Cabri-géomètre à l'école primaire, Colloque "Espace mathématique francophone", Sherbrooke, CD-Rom.
- Assude, T., Grugeon B. (2006). Développement d'ingénieries de formation des enseignants pour l'intégration des TICE. *Revue Quadrante*, volume XIII, n°2, 2004, pp.31-50.
- Assude T, Grugeon B, Laborde C, Soury-Lavergne S (2006), *Study of a teacher professional problem: how to take into account the instrumental dimension when using Cabri-geometry ?*, Etude ICMI 17 « Digital Technologies : Rethinking the terrain », Hanoi, Vietnam
- Chevallard Y (1997), 'Familière et problématique, la figure du professeur', *Recherches en didactique des mathématiques*, 17.3, pp.17-54.
- Chevallard Y (1999), 'L'analyse des pratiques enseignantes en théorie anthropologique du didactique', *Recherches en didactique des mathématiques*, 19.2, pp.221-266.
- Fischbein E. (1993), 'The theory of figural concepts', *Educational Studies in Mathematics*, 24.2, 139-162.
- Guin D & Trouche L (2002) (sous la dir), *Calculatrices symboliques, transformer un outil en un instrument du travail informatique : un problème didactique*, La Pensée Sauvage, Grenoble.
- Jones K. (2000), 'Providing a foundation for deductive reasoning: students' interpretations when using dynamic geometry software and their evolving mathematical explanations', *Educational Studies in Mathematics*, 44 (1-3), 55-85.
- Lagrange J.B. (2001), L'intégration d'instruments informatiques dans l'enseignement: une approche par les techniques. *Educational Studies in Mathematics*, 43, 1-30.
- Lagrange J.-B., Artigue M., Laborde C. and Trouche L. (2003), 'Technology and Mathematics Education: A Multidimensional Study of the Evolution of Research and Innovation', in A. Bishop and alii (eds), *Second International Handbook of Research in Mathematics Education*, Kluwer Academic Publishers, Dordrecht, pp.239-271.
- Laborde C., Capponi B. (1994), 'Cabri-géomètre constituant d'un milieu pour l'apprentissage de la notion de figure géométrique', *Recherches en Didactique des Mathématiques*, 14-1.2, 165-210.
- Laborde C. (1998), 'Visual phenomena in the teaching/learning of geometry in a computer-based environment', in C. Mammana and V. Villani (eds), *Perspectives*

on the teaching of geometry for the 21st century, Kluwer Academic Publishers, Dordrecht, pp.113-121

Hoyles, C., Noss, R., Kent, P. (2004) On the Integration of Digital Technologies into Mathematics Classrooms, *International Journal of Computers for Mathematical Learning*, 9(3), 309-326

Trouche L (2005), Construction et conduite des instruments dans les apprentissages mathématiques : nécessité des orchestrations, *Recherches en Didactique des Mathématiques*, 25.1, 91-138.

Vérillon P. & Rabardel P. (1995) Cognition and artifacts. A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education* 10(1), 77-101.

INTEGRATION OF COMPUTER ALGEBRA IN AN OPEN LEARNING ENVIRONMENT

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Mathematics teachers nowadays have to deal with two challenges: to include new teaching methods and to integrate computers in their teaching. Both changes are perceived as time consuming and an additional burden. In contrast to this, the results of a design research project at the University of Duisburg-Essen show that both can be dealt with success – without extra time. Furthermore, the design provides opportunities for achieving aims in terms of contents and processes. This paper describes the main results of the PhD-project “Mathematics between construction and instruction – evaluation of a learning workshop with integrated use of CAS”), which combines quantitative and qualitative studies.

INTRODUCTION

The idea for the project “*Mathematics between construction and instruction*” came up within the framework of the teacher training organisation T³ Germany (Teachers Teaching with Technology). Through an online questionnaire, teachers have been asked about their requests and needs for teacher training. One result of this questionnaire was the teachers’ request for more material for long-term-sequences for compulsory topics to get an idea of how teaching can change when CAS is involved. On the other hand, teachers actually have to deal with the demand to change their teaching culture by using new methods and new tasks to reach standards concerning content-related and process-related competencies. These two trends (CAS use and new methods) are often seen by many teachers as independent requirements and additional burdens. In contrast, other teachers perceive those trends not as an impediment, but rather as special opportunities to achieve aims both in terms of contents and processes. Following the idea of design science, material for teaching has been created as the basis for research to investigate whether the combination of the two trends is an impediment or an opportunity.

At first the theoretical framework will be described followed by the design of the teaching material to give a short overview of what it looks like. Then, the methodology will be presented and finally the results and conclusions of the research project.

THEORETICAL FRAMEWORK AND RESEARCH QUESTION

The whole project can be described by design science (Wittmann, 1995) in the sense of creating material, evaluating it in the complexity of every day teaching, and further developing the material. According to the aim of the project, to show whether such a way of teaching supports the learning of mathematics, it was necessary to involve a

theory about how the process of learning mathematics can be described. For this, the theoretical framework involved basically the notion of the epistemological triangle. Beside that further aspects of different theories – especially the integration of different representations and the ideas of open classroom teaching – helped to improve the design of the teaching material.

Steinbring (2005) describes mathematical conceptualising and meaning as mediations between signs or symbols and a suitable reference context. The connection between signs, reference context and the mediation between both, which is influenced by the epistemological conditions of the mathematical concept, can be represented in the *epistemological triangle*. Steinbring points out that the reciprocal actions between the “points” of the triangle and the structures of each “point” must be actively produced by the student in the interaction with others and with the teacher. The whole process of conceptualisation is then a process which can be described by series of changing triangles initiated by communication. This model takes into account the importance of the individual perspectives and experiences which influence the development of mathematical knowledge. Mathematics teaching has to initiate these processes for every student to enable a deep learning and understanding of the related topics. Coming from this theoretical basis, the design of new teaching material has to specify the necessities of the single topic, has to enable a lot of communication processes in the classroom and has to involve technology to support the learning and understanding of the concrete mathematical topic.

Calculus traditionally focuses on mastery of symbolic calculations (Tall, 1997). Tall points out how important it is to involve different kinds of mathematical representations to overcome the reduction on symbolic orientation. The *integration of different representations* allows different approaches according to the “Rule of the three” – symbolically, numerically, graphically (Hughes Hallett, 1991). For Germany Borneleit et al. (2001) also realize that Calculus teaching has a strong focus and reduction on symbolic manipulations and skills. They claim that teaching has to integrate other approaches like fundamental ideas, building a network between different aspects of one topic and enable students to develop process-competencies like arguing, structuring, investigating, modelling. For the topic “investigation of polynomial functions” they want to achieve that students are able to analyse the characteristics of a function in a competent way instead of fulfilling skills and routines without understanding. Furthermore, they see in the integration of technology and open learning arrangements a meaningful way to support this change.

Dewey (1907) and Parkhurst (1992) pointed out the value of *open classroom organization*, where students are free to select their own way through given tasks. They realized the idea of classroom organized like “laboratories”. Such an organization supports a teacher when coming along with the challenge of instrumental orchestration (Guin and Trouche, 2002) when every student has the technology available. Teachers have quite often a certain ease in their teaching

practice which is disturbed or even trivialized by CAS. Therefore, it is necessary to give them an idea of how a change could look like.

The research question

Evaluating “real-life” lessons in a multifaceted and long-term arrangement requires both a clear focus and a complex design. The basic hypothesis of the project was that pursuing curricular requests and using technology in an open classroom organisation are not contradictions but goals that can be obtained simultaneously. This should be shown by construction of an example. But how can one find out whether a new way of teaching is a meaningful one or not? What are the criteria to find out the “value” of a teaching approach? To come along with this challenge the focus of the research has been on the students’ activities which are initiated by this way of teaching. Therefore, the focus is not on the teachers’ but on the students’ activities. Thus the central question of the study is: To what extent is this learning arrangement suitable for simultaneously pursuing aims both in terms of content and process? What cognitive and metacognitive activities are promoted in such an open learning arrangement with an integrated use of CAS?

THE DESIGN OF THE TEACHING MATERIAL

Concerning the initial idea of the project the design of the teaching material was guided by two main intentions:

- When CAS is always available for the students the tasks and the teaching must change – what is a meaningful way to do this?
- How must tasks and teaching organization change to realize a constructivist approach which leads students to competencies in terms of content and process?

To follow these intentions a mandatory topic was chosen which is strongly influenced when CAS is always available: Investigation of polynomial functions.

The whole organisation of the teaching is like a workplace or a circle with different ways and possibilities of approaching the topic. This kind of classroom-arrangement is called a “Lernwerkstatt” (translated: learning workshop, laboratory) and follows the ideas of *laboratories* brought up by Dewey (1907) and Parkhurst (1992).

The students of the 11th grade (approximately 17 years old) receive a work folder on paper (Barzel et al., 2003) with a set of worksheets (or set of “modules”) that they deal with independently in groups of 4-6 students for approximately 6 weeks. Supplementary material concerning individual modules is laid out in the classroom. The students have to document their learning process in a kind of diary or portfolio. CAS is always available, on computers (such as *Derive*) or on handhelds (such as *TI-89/92*, *V-200*).

The material is meant to be used as an introduction into the aspects of the investigation of polynomial functions including differential calculus (like slope,

zeroes, extrema, inflection point). The only previous knowledge the students must have is the idea of derivation. The aim is that students learn how to investigate a function including different criteria and to gain sovereignty in doing this.

A variety of different types of tasks is involved in the material to evoke different kinds of student activities, for example: tasks which demand open ended approaches; tasks which stimulate discussions between the students; tasks which initiate flexibility between the different representations in different modules to address different learning types and tasks which integrate students' own experiences and experiments. At several points in the learning workshop, a comparison of different types of representation of a function (graph, term, table, situation) is taken as the theme and the advantages and disadvantages are discussed (see fig. 1, question on top).

According to the stimulation of different cognitive activities among students, the tasks initiate as well a variety of different ways to use the CAS. There are special assignments that require specifically the use of CAS and other assignments for which solutions are possible without this technology, but where CAS can be helpful. CAS is used for generating examples, calculating (solving equations, systems of equations, determining derivatives and single values), checking calculations and ideas, visualising certain aspects (Doerr and Zangor, 2000).

The following examples of tasks should give an impression of the variety:

- In some tasks functions are given with concrete analysis assignments. These are performed for example in module E (fig. 1, question on top). Three different functions are given, one as a graph, one as a table, and one as a formula. Pupils have to recognise the properties of an extremum.
- In some tasks connections have to be found by investigating graphs of functions and their derivatives.
- In some tasks statements have to be discussed. If you concern yourself critically with a predetermined statement, you reflect on and interlink knowledge (fig.1, question on the right, bottom).
- In some tasks functions are given with concrete analysis assignments. These are performed for example in module E (fig. 1, question on top). Three different functions are given, one as a graph, one as a table, and one as a formula. Pupils have to recognise the properties of an extremum.

You can see three functions in different representations. Determine the local extrema and try to define the concept "local extrema". What are the benefits and problems of the different representations?

Function 1:

Function 2:

$$f(x) = x^2 + 2$$

mit $-2 \leq x \leq 2$

Function 3:

x	y
0	13,5
1	5,94
2	1
3	-1,69
4	-2,5
5	-1,81
6	0
7	2,56
8	5,5
9	8,44
10	11
11	12,81
12	13,5
13	12,69

Find a calculation to determine local extrema. Use this calculation for the functions given by the following equations.

Check by plotting the graphs.

$$f(x) = \frac{1}{3}x^3 - x \quad \text{and} \quad g(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2$$

Why is the adjective „local“ important?

„If the first derivative is 0, then there is a local extremum!“

Discuss this statement and correct it if necessary.

Figure 1: One module as an example: Module E – Extrema

- In some tasks connections have to be found by investigating graphs of functions and their derivatives.
- In some tasks statements have to be discussed. If you concern yourself critically with a predetermined statement, you reflect on and interlink knowledge (fig.1, question on the right, bottom).
- In some tasks students collect experimental experience. One module encourages trials with a sonic motion detector (CBR – “Computer Based Ranger” connected to a TI calculator), where movements are recorded indirectly as a time-distance or time-velocity diagram. Cognitive discussions are linked with concrete experience in order to facilitate comprehension of the new contents.

An overall reflection of the workshop is finally performed by preparing posters for a final presentation.

To give teachers an idea of how to use the material in their classroom teaching, an introductory booklet with the main ideas and recommendations for realising the workshop in their own teaching serves as a guideline. This booklet also contains additional material for displaying in the classroom (e.g. two games). In order to

evaluate the learning process, the teacher has several possibilities: s/he can assess the group work by judging the individual student's participation and engagement during the group work and by assessing the student's presentation of results (by poster or other visualisation). In addition, the teacher can check the learning journals of the individual student and, of course, the results of the students' final written test.

METHODOLOGY

According to the multifaceted learning arrangement, a complementary research design collecting qualitative and quantitative data was chosen. The qualitative part is an interpretive study based on video tapes. The quantitative part is an experimental large-scale study. A complementary research design was chosen which evaluated the teaching in a qualitative way and a quantitative way. In the pilot phase, the teaching material was initially tested by six teachers, discussed with those teachers, and afterwards further developed and published (Barzel et al., 2003). Furthermore, the questionnaires for the subsequent quantitative study also were piloted.

The qualitative part of the study

For the qualitative assessment, a class was monitored during the teaching with the learning workshop. Mainly the lessons were recorded on video, interviews with students and the teacher were conducted, students' learning journals were analysed, and examination papers of the experimental class were studied in comparison to examination papers of parallel classes. Portions of the material were coded and analysed and assessed according to the guidelines of the Grounded Theory (Strauss and Corbin, 1996) – these include transcripts of teaching and the final examination paper of the experimental group (and the parallel classes as comparison groups).

In this process, some lessons were chosen for analysis and interpretation showing a range of different activities inside small group discussion. As an instrument for interpretation, the epistemological triangle of Steinbring (2005) has been used. For the specific purpose of this study the epistemological triangle as an instrument is supplemented by the perspective of cognitive activities. With this modified instrument the interactions between the students have been interpreted regarding the process of individual conceptualizing and the process of interacting in the communication with the others.

The quantitative part of the study

This part includes a post-survey (one for students and one for teachers) and a comparative post-test. For quantitative assessment, it was possible to recruit a total of 45 teachers (approximately 1200 students) in the school year 2003/04 in order to work on the topic with the aid of the learning workshop. This was the basis for an experimental large-scale study to look across schools to support broad generalisation. Both teachers and students were asked to complete a post-survey, and 578 students and 17 teachers gave feedback in this manner. These questionnaires enquired about

individual attitudes to the work in the learning workshop, among other aspects, via statements that needed to be classified on a Likert scale.

The participating classes took part in a final test.

RESULTS

The results of this complex research project are manifold and could give detailed information for many aspects. Let me point out here some main results of both parts – the qualitative and quantitative study.

The qualitative study

The observation of group work interaction over a period of time (six weeks) and the analyses of single sequences show that students develop a variety of **metacognitive strategies** to come along with the challenge of such a long-period self-dependent work. They use new approaches to master difficulties, they look up for further information in books or reflect their learning process and try to find out reasons for their difficulties.

The analysis of two transcripts has revealed the **value of the informal speech** in a small group of students. Students are not only able to explain and correct each other, but they show clear steps in their conceptualization. These steps are quite often initiated by listening to another student and by understanding his/her different approach. For example, when analysing a graph you can observe two different ways to do it – one way is to focus special points (such as extremum, zero), the other one is a more dynamic one to follow the slope. As soon as the students have recognized the two ways, they use and reflect them as additional strategies to solve the problem.

CAS is used within this learning environment adapted to the own learning process – quite often for short time-periods – for generating examples, controlling and visualizing. It could be observed that students use CAS in a spontaneous way to foster their own argumentation. For example, one student said to another when looking for the maximal number of zeroes by a cubic function:

Student: Wo habt ihr denn da bitte schön zwei Wendepunkte oder zwei Extrempunkte – hat die nämlich gar nicht

(Translation: Please, where has this one two inflection points or two extrema – it doesn't have it (*showing the screen of her TI-89 with the graph of x^3*))

This impulse led to a discussion about the meaning of “maximal” in this context.

Students of the experimental group gain more **flexibility in analysing graphs and equations** than students who are taught in a traditional way. This is a result of analysing students' solution of an assignment in a final test within the framework of the qualitative study. The test was written by the experimental group and three parallel classes as comparison groups from the same school. The test was for all students the same official central examination paper. All the students' solutions

without the respective teacher's corrections were available to the group of researchers. It was apparent while correcting the answers to the task in fig.2 that the students in the experimental group often began with the answer phrase and supplied the justification subsequently, unlike students in the comparison groups who began with justification and ended with the answer phrase. These and other characteristics were consequently quantified in order to detect possible trends. The first impressions have been confirmed by this and the examination of these characteristics provided possible reasons for this phenomenon. The working direction in the comparison group was predominately from term to graph. In contrast to this the working direction in the experimental group was more frequently from graph to term or moved back and forth between the two representations.

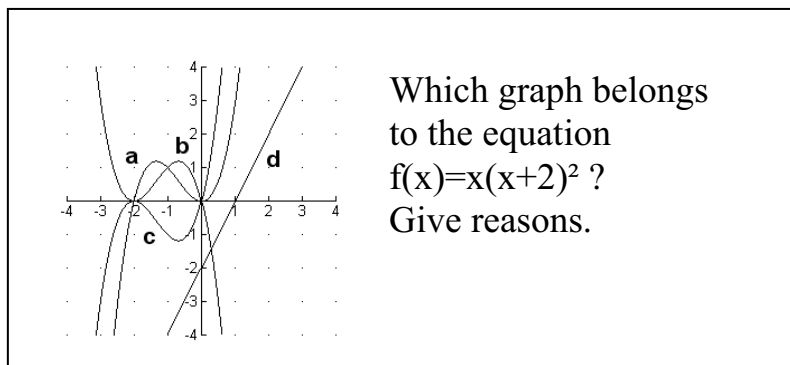


Figure 2: Task of a final test

The quantitative study

A post-survey for teachers and pupils to collect the experiences of the actors and a final comparison test have been the two parts of the quantitative study. 578 students and 18 teachers answered to the survey.

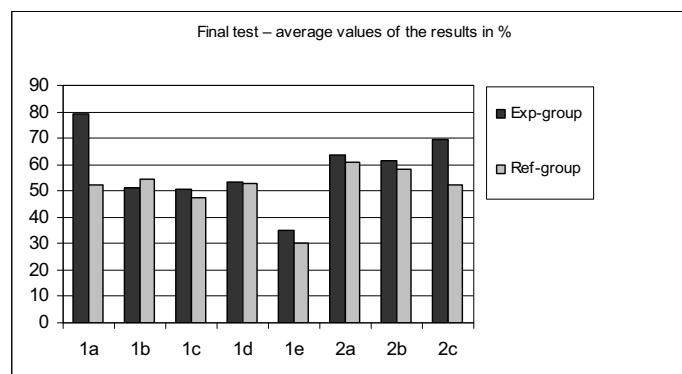


Figure 3: Results of the final test

The final test consisted of two questions which were adopted from former central comparative examination papers in North Rhine-Westfalia in order to select a requirement imposed from outside. The results of the test show that this learning environment with an integrated use of CAS does not require **extra teaching time**. The objectives can be achieved in the same time as by a traditional way of teaching.

This can be concluded by the results of this final comparison test (Experimental group: 462 students, Comparison group: ~11000 stud, see fig. 3).

The results of the post-survey show that both teachers and pupils highly appreciate this kind of learning, especially the weak students give a positive feedback. Reasons for this can be drawn out from the answers to several questions in the survey. Weak students point out that the possibility of having enough time to clear up difficulties in an informal speech with others is a great advantage. Concerning the role of CAS in this learning environment, the answers are very clear. Students and teachers highly appreciate the use of CAS – 72% of the students look favourably on its use and do not want to miss it. CAS was used in some classes on handhelds (*TI-89/92 or V-200*) and in some on a PC (*Derive*). It is interesting that the part of those students who evaluated CAS favourably was higher in the handheld group (78%) than in the PC-group (65%). The availability seems to be an important issue.

CONCLUSION

The results obtained with regard to the research question allow a positive assessment of this learning arrangement, it is suitable for simultaneously pursuing aims both in terms of content and process. A big variety of different metacognitive and cognitive activities with an epistemic gain can be observed thanks to the combination of an open learning arrangement and the integration of CAS. Analysing and structuring of equations, graphs and words and as well explaining, arguing and reasoning can be observed and are pointed out from students themselves as highly promoted activities.

The results selected should not obscure the fact that problems with an educational arrangement of this type might occur, particularly when the necessary openness in teaching is new and unfamiliar for both the learner and the teacher. Such a form of teaching requires abilities in a special way, specifically on the part of the teacher, in order to be able to use the variety and diversity productively in teaching.

The observation that a variety of diverse cognitive activities are stimulated is a cause for hope that content and process aims are equally pursued in this manner. Moreover, the results show not only a range of interesting fields for further research but also interesting aspects for exchange with teachers within the framework of in-service-training. The learning workshop is already and will be subject of teacher training – not only as an introduction of the material or learning how to use a certain CAS but especially in the way to discuss selected results of the research project with teachers as a basis of reflection and development – of the material and of the own way of teaching.

The way gives hope that the changes induced by such teaching material seems to be acceptable for teachers and seems to help them to modify their praxeologies in teaching. Providing the teacher not only with students' activities but also with adequate teaching information and additional materials might help them to feel more

confident. There is an immense request from teachers to obtain more similar material for other topics since the time this material has been published.

REFERENCES

- Barzel, B. & Fröhlich, I. & Stachniss-Carp, S. (2003). *Das ABC der ganzrationalen Funktionen, Schülerheft & Lehrerheft*. Stuttgart: Klett
- Barzel, B. (2006). *Mathematikunterricht zwischen Konstruktion und Instruktion. Evaluation einer Lernwerkstatt im 11.Jahrgang mit integriertem Einsatz von Computeralgebra*. Dissertation, University of Duisburg-Essen
- Borneleit, P. & Danckwerts, R. & Henn, H. & Weigand, H. (2001). Expertise zum Mathematikunterricht in der gymnasialen Oberstufe. *JMD* 22 (1), 73 – 90
- Dewey, J. (1907). *The School and Society: being three lectures by John Dewey supplemented by a statement of the University Elementary School*. Chicago: University of Chicago.
- Doerr, H.M. & Zangor, R. (2000). Creating meaning for and with the graphing calculator. *Educational Studies in Mathematics* 41, 143-163
- Guin, D. & Trouche, L. (2002). Mastering by the teacher of the instrumental genesis in CAS environments: necessity of instrumental orchestration. *ZDM*, 34 (5), 204 – 211
- Hughes Hallett, D. (1991). Visualization and Calculus Reform. (In: Zimmermann, W. & Cunningham, S. (Eds.), *Visualization in Teaching and Learning Mathematics*, MAA Notes No. 19, 121-126.
- Parkhurst, H. (1922). *Education on the Dalton Plan*. New York: E.P. Dutton and Company
- Steinbring, H. (2005). *The Construction of New Mathematical Knowledge in Classroom Interaction: An Epistemological Perspective*. Heidelberg: Springer
- Strauss, A.L. & Corbin, J. (1996). *Grounded Theory: Grundlagen Qualitativer Sozialforschung*. Weinheim: Beltz
- Tall, D. (1997). Functions and Calculus. (In: Bishop, A. J. et al (Eds.). *International Handbook of Mathematics Education*. pp. 289-325) Dordrecht: Kluwer.)
- Wittmann, E.Ch. (1995). Mathematics Education as a “Design Science“. *Educational Studies in Mathematics*, 29/4, 355-374

TEACHING ANALYSIS IN DYNAMIC GEOMETRY ENVIRONMENTS

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In this paper we present instructional activities for the concepts of limit, derivative and definite integral which were designed in the course of a European Comenius 2.1 project. The learning environments are designed so that they approach intuitively, through multiple and interconnected representations, the above mathematical concepts in ways that are consistent with the mathematical theory. Additionally, they take into account students' previous knowledge as well as topics that previous research suggests causing difficulty in calculus courses. The material we present here makes use of dynamic geometry software that offers a function editor / sketch environment as well as tools appropriate for Calculus instruction. Some elements of teachers' feedback in this material are presented.

INTRODUCTION

Functions and Calculus have a wider field of applications in other disciplines and constitute a basic part of the mathematical curriculum of secondary education. In many countries good student performance in pre-Calculus is necessary for university entry. At the same time, as several studies show, the majority of students face serious problems in understanding the concept of function as well as the basic Calculus concepts (for example see Harel, Selden & Selden, 2006).

The work presented in this paper originates in a three-year project called *CalGeo* (Teaching Calculus Using Dynamic Geometric Tools). Amongst the objectives of this project is the design of an in-service teachers' education programme based on learning environments suitable for teaching functions and Calculus in upper secondary education using dynamic geometry tools. The programme focuses on the following topics: introduction to infinite processes, limit, continuity, derivative and integral¹. For each topic the training material includes documentation that raises mathematical, historical and didactical / pedagogical issues and a set of proposed activities. According to the project, the produced material has to be tested in a pilot teachers' training programme as well as in real classroom conditions in each of the participating countries. By the time of the writing of this paper the teachers' training programme has been finished in Greece without any classroom application.

In what follows we first articulate the theoretical assumptions of the project. Then we describe the dynamic environment within which the activities were developed

¹ The countries involved in this project are Greece, Cyprus, England and Bulgaria. The produced material refers to grades 11 and 12 and it is adapted to the Calculus curricula of the participating countries.

accompanied by some results of the pilot training programme applied in Greece. Finally we present some issues for discussion that emerged from this work.

THEORETICAL ASSUMPTIONS

The role of representations in mathematics education has emerged in recent literature as fundamental aspect in facilitating students' construction of mathematical concepts as well as their problem solving abilities (Janvier, 1987; Schoenfeld, 1994; Zimmermann & Cunninham, 1991). Additionally, the technology-based multiple linked representations influence the students' mental constructions of Calculus concepts. For a deeper conceptual understanding of Calculus, instructions should be focused not only on the use of algebraic representations but additionally should take into account the geometric and intuitive representations of the corresponding mathematical objects as well as the interaction among these multiple representations (Kaput 1994). This mode of representing mathematical concepts is gaining more strength due to the advances in computer technology and the development of dynamical mathematical software (Habre & Abboud, 2006).

Mere experience of different representations however does not entail deeper conceptual understanding. All the difficulties could not be explained in terms of a lack of interconnection between representations. Sometimes students' intuitive ideas or informal-spontaneous perceptions influenced by their everyday experience and previous mathematical knowledge block their path towards knowledge acquisition (Cornu, 1991, Fischbein, 1987). These ideas act as epistemological obstacles narrowing students' understanding (Sierpiska, 1994).

The design of a multiple representations learning environment appropriate for Calculus concepts needs to take the considerations expressed by the above literature into account. The activities we describe and discuss in the following section aimed to do so. In this paper we discuss the activities revolving around the definitions of limit, derivative and definite integral.

LEARNING ENVIRONMENT / ACTIVITIES

Each of the following activities was designed in order to be used towards the introduction of a new Calculus concept at upper secondary education level (grade 11 and 12). The activities make use of students' previous knowledge in the following two ways:

- They offer problem solving situations in which previous knowledge will turn out inadequate (e.g. the calculation of the instantaneous velocity or the area defined by a function graph and the x' axis cannot be done by finite processes as in average velocity or in areas of rectilinear geometrical shapes, respectively).
- They offer the opportunity to explore alternative and generalisable aspects of an already known concept (e.g. the tangent line of the circle as the limiting position of secant lines).

These problems are either historical problems (e.g. Archimedes' calculation of an area or the properties of the tangent line in Euclid's Elements), or modelling of real problems (e.g. problems of calculation of surfaces, motion problems).

The learning environments are designed in order to:

- Approach intuitively the corresponding mathematical notion(s) in ways that are consistent with mathematical theory (e.g. visual representation of the ε - δ definition of the limit).
- Take into account the students' previous knowledge (e.g. students' knowledge about circle tangent towards their introduction to derivative).
- Take into account the topics which have proved to be a source of learning difficulties in calculus courses (e.g. difficulties in understanding definitions and their geometric interpretations).
- Embody an intuitive approach of the concept to be taught (e.g. local straightness as the embodied approach of local linearity (Tall, 2003)).
- Offer multiple and interconnected representations of the same concept (e.g. the definite integral as the measurement of an area, as the limit of a sum, as new symbolic expression etc.).

The environments we have developed utilise a dynamic geometry software (DGS) called EucliDraw. In addition to DGS facilities, this software offers a function editor / sketch environment as well as some tools appropriate for Calculus instruction. Indicatively, we refer to the magnification tool that can magnify a specific region of any point on the screen in a separate window. This magnification can be repeated as many times as the user specifies through a magnification factor. Other useful, for Calculus, facilities are these that can partition an interval; construct the lower and upper rectangles covering the area defined by a graph and the x ' x axis; control the number of the decimal numbers of calculations etc.

Activity on the concept of Limit

This activity starts with an instantaneous velocity problem:

Problem: A camera has recorded a 100m race. How could the camera's recording assist in calculating a runner's instantaneous velocity at $T=6\text{sec}$?

The students are familiar with the notion of average speed, through their everyday experience and their school experience in Mechanics. But, for the transition to the calculation of instantaneous velocity understanding the limiting process is essential.

The aims of this activity are the intuitive introduction to the ε - δ definition of limit of a function and the connection of numerical and graphical representations towards the clarification of the concept of the limit of a function. Research studies have shown that students often understand the notion of a limit as a dynamic process of "getting close to" a fixed point, often with the perception of "never reaching" the limit; or, as

a certain algebraic procedure to be done and not as a static concept (Cornu, 1991). Since the formal definition is in disagreement with students' dynamic, intuitive ideas of limit, this persistence of their informal ideas means that students are often unable to make sense of the formal definition. Disagreement between formal definitions and informal concepts is only one example of a situation in which a student may hold two mutually contradictory ideas and not notice a conflict. The wording of the question plays a role in the selection of the particular mental image that the student brings into play when presented with a problem. John Monaghan (1991) studied the effects of the language used in teaching and learning limits and he noticed that students should realise how everyday meanings of mathematical phrases can direct them into fallacious interpretations.

The technology-based learning environments offer the facility of graphic representations as well as arithmetic computation. Working in these environments a more balanced approach could be achieved to the concept of limit. This approach could offer the appropriate visualisation with the computer carrying out the calculations internally. In this activity the dynamic geometry based environment, already designed in a file of EucliDraw software, acts in two different roles. In the first part of the worksheet it provides numerical results. This enables students to avoid time consuming calculations. In the second part of the worksheet, it represents the numerical data graphically. The students can then visualize the convergence of function and make the transition to the ε - δ definition. In this environment the axis of the time is displayed (x 's axis) and the student can change the time t and the fixed time T . The values of the corresponding distance of the runner $s(t)$ and $s(T)$ are calculated as well as the average speed $\frac{s(T)-s(t)}{T-t}$. In the second part

of the activity, the graph of the average speed function $U(t)$ is sketched. Additionally, the ε zone, an area of length 2ε around L , and the δ zone, an area of length 2δ around T , are displayed ($L=U(T)$). A part of these two zones is coloured in red and green. The green and red colours are used not as a visual effect, but as a tool to represent verbally and graphically complex symbolic expressions.

According to these constructions this part of the graph that includes the points $(t,U(t))$ such as $t \in (T-\delta, T+\delta)$ is in the red zone if $U(t) \notin (L-\varepsilon, L+\varepsilon)$ or in the green square if $U(t) \in (L-\varepsilon, L+\varepsilon)$ (figure 1). The colours of these parts have a semiotic role: the red is the restricted and the green is the permissible area according to the definition.

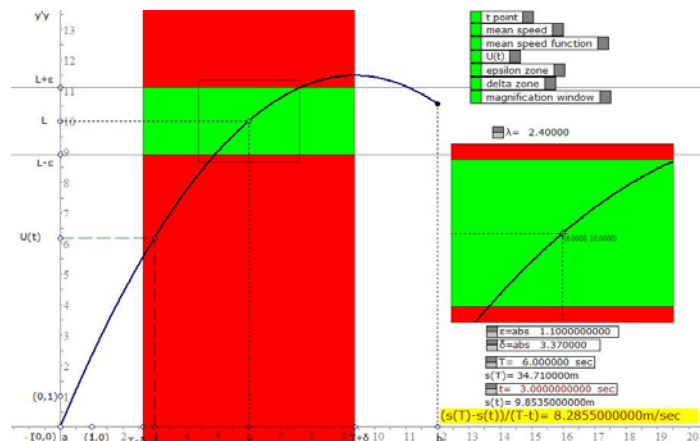


Fig. 1: A EucliDraw environment for the introduction to the notion of limit

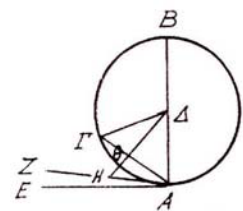
The use of the ε and δ zones in the dynamic geometry environment, gives the student the opportunity to handle in a dynamic way the basic parameters of the problem, in order to understand their relations. The students can change, following the worksheet, the values of ε and look for the appropriate values of δ such as no part of the graph to be in the red region.

The tool of the magnification is used in order to magnify a region of the point (T, L) when the values of ε are too small and the visual perspective is inaccurate.

Activity on the concept of Derivative

This activity starts with a problem inspired by Euclid's "Elements". The students are asked to check the validity of the following proposition (proposition 16 in the third book of 'Elements')²:

"The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilinear angle."



The aims of this activity are: the introduction to the definition of the derivative at a point; the introduction to the definition of the notion of the tangent line of a function curve at a point $(x_0, f(x_0))$ as the linear approximation of the curve at this point; the reconstruction of students' previous knowledge about tangent line grounded to the Euclidean geometry context to general cases of curves; the connection of the symbolic and geometric representations of derivative at a point; the recognition by the students the property of the "smoothness" of a function curve at a point and its relationship to the differentiability of this function at this point.

According to Tall (2003), the cognitive root of the notion of derivative is the local straightness. The property of local straightness refers to the fact that, if we focus close enough to a point of a function curve, in which point the function is differentiable, then this curve looks like a straight line. Actually, this "straight line" is the tangent line of the curve at this point. This property is valid in all cases of tangent lines and it could be facilitated, wherever it is possible, by the use of new technology with appropriately designed software (Tall, 2003; Giraldo & Calvalho, 2006). On the other hand the early experiences of the circle tangent contribute to the creation of a generic tangent as a line that touches the graph at one point only and does not cross it (Vinner 1991). Furthermore, students perceive relevant and irrelevant properties related to the number of common points or the relative position of the tangent line and the graph as defining conditions for a tangent line. Different combinations of these properties create intermediate models of a tangent line. This occurs through the assimilation of

² Translated in English by Heath, (1956, 2, p. 37)

new information about graph tangents in the existing knowledge about circle tangent (Biza, Christou & Zachariades 2006).

This activity starts with the notion of circle tangent. New and more generalisable properties of the tangent line are introduced such as “the best linear approximation of the curve at a point” or “the limiting position of the secant line”. These properties are approached intuitively through the local straightness by the magnification of the curve. These new properties offer an opportunity for a transition to a general definition of a tangent line of a curve and consequently to the introduction of the notion of derivative.

The activity is divided into three steps. The first one is developed in the Euclidean Geometry context. With the help of the magnification tool they magnify the region around A and they observe that the circle looks like its tangent as the magnification factor increases. The students observe that the tangent line is the limiting position of the secant lines AB as B approaches A . In the second step a transition to the Calculus context is made and students are asked to define a line as a tangent at the case of a function graph. In this step, the EucliDraw file is supposed to be already constructed. In this environment the graph of $y = \sin(x)$ is sketched as well as the point $A(x_0, f(x_0))$. In addition, there are three different buttons that hide the constructions of the following tasks of the exploration. The first one is the “magnification” button that displays the number h , the points $B(x_0+h, f(x_0+h))$ and $C(x_0-h, f(x_0-h))$ and the magnification window of a region of A related to a magnification factor equal to $1/k$. As the h decreases the magnification factor increases and the points B and C are moving closer to A . The second one is the “secant lines” button that displays the secant lines AB and AC . Finally the third one is the “slope” button that displays the slopes of the lines AB and AC (figure 2). In this step students, through different tasks on a worksheet, decrease the number h , observe what has happened with the secant lines and their slope, and they are introduced to the definition of derivative. In the third step students work in the same environment by changing the function to the $y = |\sin(x)|$ that is not differentiable at the points in which the graph meets the x -axis. The students, through the observation on the graph at these points, are introduced to the property of the “smoothness” of a function curve at a point and its relationship to the differentiability.

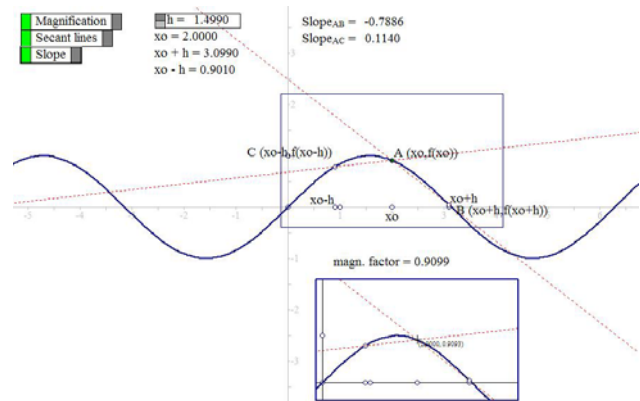
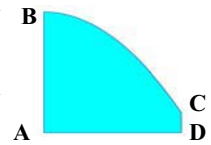


Fig. 2: A EucliDraw environment for the introduction to the notion of derivative

Activity on the concept of Definite Integral

This activity starts with a problem of a parabolic area calculation in order to introduce the students to the notion of Riemann integral:

We are looking for a way to calculate the area of the semi-curved region $ABCD$, which is bounded by three segments AB , AD (x' -axis), DC , and a parabolic segment BC , coming from the graph of a quadratic equation.



Successive calculations of both the upper and the lower Riemann sums are realized in a dynamic geometry environment, for different partitions. Both notions of bound and approximation are essential in the whole process.

The aims of this activity are: the introduction to the notion (and even the definition) of the area of a semi-parabolic region of the plane, through a “natural”, multi-representational way; the intuitive use of Riemann sums in relation to both notions of bound and approximation; reconstruction of the students’ previous beliefs about the notion of measurement, and manipulation of arithmetic, symbolic and geometric representations of the same concept.

Research findings on pre-calculus teaching show that many students, though capable to calculate derivatives or integrals, they do not develop conceptual understanding (Orton, 1983). The activity has the intention to reverse this sequence, and introduce conceptual understanding before or at least in parallel with procedural understanding. The initial problem is the calculation of the area of a semi-parabolic region which can not be achieved by the traditional methods of area calculation. Riemann sums are introduced and used in a “natural” way to extend the traditional notion of area measurement which is useless in this case. Realization and connections with the geometric representation of the whole situation with area counters are supported by the EucliDraw environment. Dynamic tools as the control parameters for the number of covering rectangles and the magnification tool are inevitably used so as to help students understand the meaning of the questions.

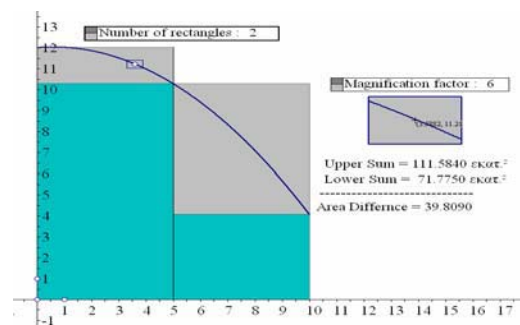


Fig. 3: A EucliDraw environment for the introduction to the notion of definite integral

In this activity the EucliDraw file is supposed to be already constructed. In this environment a parabolic graph is sketched in the domain $[0,10]$ as well as the following constructions: the covering rectangles, both above and under the curve; the parameter n that controls the number of these rectangles; the “upper” and “lower” sums of the rectangles’ areas above and under the curve respectively (are called Riemann sums) and the area difference of these sums. The magnification tool is used to magnify a region of a specific point of the curve as many times as the magnification factor indicates in order to give some sense of the accuracy of covering the parabolic area with rectangles (figure 3).

The design of the activity has taken into account students' previous knowledge about area calculation of rectilinear geometrical shapes. Firstly, the students are asked to think about simpler problems of the same type (e.g. rectilinear geometrical shapes of the plane), having an area which can be easily calculated and to try to apply methods that are already familiar with to the case of parabolic area. Then the students, by using the above described environment and increasing the number of covering rectangles, make, step by step, better approximations for the area in question. If the number of covering rectangles is too big to give a visual perception of the difference between the upper and lower areas, the magnification tool clarifies the visual representation. Students try different values of n , observe the graphic and numeric changes, keep the measurements in a table of their worksheet, and make conjectures based on their observations towards the construction of their knowledge about definite integral.

TRAINING PROGRAMME

The participants of the pilot training programme were 18 Greek mathematics teachers who currently teach in grades 11 and/or 12. Most of the participants had basic computational skills but only a few of them were familiar with mathematics educational software (especial DGS like Geometer's Sketchpad and Cabri) and its application in teaching practices. An introductory course on EuclidDraw software was included in the programme. Although the evaluation of the training programme is still in progress, we can provide some of the teachers' comments and reactions during the programme for each of the activities.

Concerning the limit activity, some of the teachers noticed that the visualization of the effects that ε and δ parameters cause would be helpful, while some others expressed their scepticism on whether the use of this activity will actually assist in overcoming the difficulties they face in their teaching practice (although Greek students are not examined on the ε - δ limit definition, they are presented an informal intuitive introduction to it). In the derivative activity, some teachers commented that the use of Euclid's proposition may contribute to the students' understanding. However, they doubted about the efficiency of students' involvement to *horn-like* angles (defined by the circumference and a straight line). The integral activity was familiar to the teachers as they often use similar problems in order to introduce the notion of integral. Most of the teachers found the dynamic manipulation of the parameters in the environment very useful. Additionally, some of them proposed problems alternative to those of the calculation of the area of a semi-parabolic region.

CLOSING COMMENT

In the wake of recent and rapid technological developments the most challenging question regarding the teaching of Calculus in ways that benefit from these developments is how we devise tools that approach intuitively the mathematical ideas

of Calculus while being consistent with mathematical theory and while taking account the Calculus-related learning issues that mathematics education research has repeatedly highlighted. According to Ferrara, Pratt and Robutti (2006):

... the big revolution in teaching mathematics with technologies was the introduction of dynamicity in software: A dynamic way to control and master the virtual objects on the computer let the student explore many situations and notice what changes and what does not. And the mathematics of change is the first step on the road of calculus. We can intend change at a numerical level, as well as the graphical or symbolic level. (p.257)

In this paper we presented three different activities designed for the introduction to the definitions of limit, derivative and integrals that utilise the above described dynamicity of technology based environments. These activities are a part of a larger piece of work currently in progress. As until now these activities have not been tested in real classroom conditions we wish to confine the aims of this paper to the description of this material and our attempt to integrate the technological progress mentioned above with theoretical underpinnings and concerns made by the relevant mathematics education research. Additionally, we provide some elements of teachers' feedback for the purpose of discussion prior to application.

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REFERENCES

- Biza, I., Christou, C., & Zachariades, T. (2006). Students' thinking about the tangent line. In J. Novotna, H. Moraova, M. Kratka & N. Stehlikova (Eds.), *Proceedings of the 30th PME International Conference, 2*, 177–184. Prague, Czech Republic.
- Cornu, B. (1991). Limits. In D. O. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 153–166). Dordrecht, The Netherlands: Kluwer.
- Ferrara, F., Pratt, D., & Robutti, O. (2006). The role of technologies for the teaching of algebra and calculus. In A. Gutierrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp. 237–273). Rotterdam, The Netherlands: Sense Publishers.
- Fischbein, E. (1987). *Intuition in Science and Mathematics: An Educational Approach*. Dordrecht, The Netherlands: Reidel.

- Giraldo, V., & Calvalho, L. M. (2006). Generic organizer for the enrichment of the concept image of derivative. In J. Novotna, H. Moraova, M. Kratka & N. Stehlikova (Eds.), *Proceedings of the 30th PME International Conference*, 3, 185–192. Prague, Czech Republic.
- Habre, S., & Abboud, M. (2006). Students' conceptual understanding of a function and its derivative in an experimental calculus course. *Journal of Mathematical Behavior*, 25, 57–72.
- Harel, G., Selden, A., & Selden, J. (2006). Advanced Mathematical Thinking. Some PME Perspectives. In A. Gutierrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp. 147–172). Rotterdam, The Netherlands: Sense Publishers.
- Heath, T. L. (Ed.). (1956). *The thirteen books of Euclid's Elements* (Vols. 1-3). New York: Dover Publications.
- Janvier C. (Ed.). (1987). *Problems of Representations in the Teaching and Learning of Mathematics*. Hillsdale, NJ, USA: Lawrence Erlbaum.
- Kaput, J. (1994). Democratizing access to calculus: New routes to old roots. In A. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 77–156). Hillsdale, NJ, USA: Lawrence Erlbaum.
- Monaghan, J. (1991). Problems with the language of limit. *For the Learning of Mathematics*, 11(3), 20–24.
- Orton, A. (1983). Students' Understanding of Integration, *Educational Studies in Mathematics*, 14, 1–18 .
- Schoenfeld, A. (Ed.). (1994). *Mathematical Thinking and Problem Solving*. Hillsdale, NJ, USA: Lawrence Erlbaum.
- Sierpinska, A. (1994). *Understanding in Mathematics*. London, Washington, D.C.: The Falmer Press.
- Tall, D. O. (2003). Using Technology to Support an Embodied Approach to Learning Concepts in Mathematics. In L. M. Carvalho & L. C. Guimaraes (Eds.), *Historia e Tecnologia no Ensino da Matematica*, (1st vol., pp. 1–28), Rio de Janeiro, Brasil.
- Vinner, S. (1991). The role of definitions in the teaching and learning of Mathematics. In D. O. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 65-81). Dordrecht, The Netherlands: Kluwer.
- Zimmermann, W., & Cunninham, S. (Eds.). (1991). *Visualization in Teaching and Mathematics*. Washington, DC, USA: Mathematical Association of America.

ONLINE RESOURCES IN MATHEMATICS: TEACHERS' GENESIS OF USE

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The long-term objective of our research is to develop the instrumental approach for teachers. A first step, presented in this paper, is to observe stable behaviours of teachers using internet resources in mathematics. We retain the scenarios as indicators of the genesis processes. We propose a scenario taxonomy taken from categories elaborated by computer sciences specialists and complemented to take into account didactical aspects. The descriptions provided by teachers permitted to observe an evolution of their scenarios elaboration's practices.

INTRODUCTION

We present here the work of a team comprising two mathematics education researchers, two teacher trainers and five teachers, and belonging to a more general research project termed GUPTEN (which holds for, in French: Genesis of Professional Uses of Technology by Teachers, project directed by Jean-Baptiste Lagrange).

This project aims at studying, in particular, the way teachers take over a new technological tool. The word "Genesis" in GUPTEN stems from the instrumental approach, a theoretical frame encompassing elements both from cognitive ergonomics and from the anthropological theory of didactics (Chevallard 1999) and developed within mathematics didactics to study issues related with ICT. Rabardel (1995) stresses the difference between an artifact, which is a given object, and an instrument. The instrument is a psychological construct; it comprises the artifact, and the schemes and techniques the user develops while using it (Guin and Trouche 2002). The building process of the instrument is called "the instrumental genesis".

We study here specific technological resources: e-exercises bases (shortened EEB in what follows). These resources consist of exercises classified according to their mathematical content, to their difficulty, and/or to the mathematical tools they require. These exercises are associated with an environment which consists of suggestions, correction, explanations, tools for the resolution of the exercise, score etc. (for more details about the possible features of an e-exercises resource, see Cazes and al. 2005). We consider these resources as artefacts, likely to become instruments for the teacher through an instrumental genesis process. Our aim is to observe and analyse this process. EEB can also become instruments for the students (Gueudet 2006); we do not study that aspect here. Applying the instrumental approach to teachers requires a specific theoretical work. We retained as a starting point the notion of scenario. We present in part 2 these theoretical choices. In part 3 we expose methodological elements: the data collected, the main EEB used by the teachers and

the description grid of the scenarios. Part 4 is devoted to our main results and first observations of scenarios evolutions, interpreted as part of the genesis process.

THEORETICAL FRAME: INSTRUMENTAL GENESIS AND SCENARIOS

Several research works study the complex practices of teachers with ICT (Monaghan 2004). Trouche (2004) introduces the notion of instrumental orchestration, which is associated with the instrumental approach and takes into account the teacher's action with ICT. But the orchestration aims at analysing how the teacher manages the students' instrumental genesis; it does not describe the teacher's own genesis.

Analysing teachers' instrumental genesis requires a specific theoretical reflection. It means to describe schemes, or techniques, considered as the visible part of the schemes developed by teachers (Guin and Trouche 2002). A first step is thus to observe if the teachers develop stable behaviours with the EEB, likely to be organised by an underlying scheme.

So we need to answer the following question: what does it mean to observe and describe teachers' practices with an EEB? For this purpose, we used the notion of scenario. Trouche (2004) defines a scenario in use as the organisation of a situation in a given environment. The scenario encompasses the management of the situation itself, and the management of the artifacts (orchestration). We consider that an EEB requires a specific approach because it can intervene also in the management of the situation.

Therefore we turned to the notion of scenario studied by computer science specialists, notably Pernin and Lejeune (2004), whose work is inspired by the Instructional Management Systems Learning Design (IMS LD) (Koper 2001) concepts and methods. We needed to adapt their approach, because we aim at describing and analyzing teachers' practices and not at developing technology enhanced learning techniques for them. We now present the theoretical frame elaborated by Pernin to study scenarios and the way we use it.

For Pernin and Lejeune, a learning scenario « represents a description, made a priori or a posteriori, of the progress of a learning situation at a given level, or learning unit, whose goal is to ensure the appropriation of a precise set of knowledge. A scenario describes roles, activities and also knowledge resources, tools and services necessary to the implementation of each activity. » (Lejeune and Pernin 2004). In our study, scenarios' designers are the teachers and users are the students. Pernin and Lejeune propose a list of six criterions to elaborate a taxonomy of scenarios.

Purpose: a scenario can be **scheduled** (developed a priori in order to define a learning situation) or **descriptive** (describing what actually happened during the learning situation). In our study, the scenarios can be scheduled or descriptive.

Granularity: Pernin and Lejeune introduce three granularity levels: the level of an elementary activity, the level of a sequence or composite activity and eventually the level of a structuration unit. Teachers may plan to use ICT for a one hour session to

pursue a precise mathematical goal or may construct a sequence (a set of sessions with a same mathematical content).

Constraint: a **constrained** scenario describes precisely the activities to be realized, leaving a reduced initiative to the actors of the learning situation. An **open** scenario let them organize their progression; the main lines of the activities to be performed are described. At last, an **adaptable** scenario is an open scenario which can be modified by the actors of the learning situation. We can analyze the teacher's scenarios with this criterion: are they constrained or are students free to thumb through the EEB? For us, a scenario can never be adaptable because a student doesn't have the possibility to modify a scenario established by a teacher.

Personalization: a scenario can be **generic** (its execution is always the same) or **adaptive** (taking into account the user's profile and allowing execution of personalized scenarios). We use this criterion to sort scenarios depending on whether teachers chose to differentiate their students a priori, for instance with personalized menus.

Formalization and reification: we only consider here **informal scenarios**, empirically created by teachers (and not **formalized scenarios** using an educational modelling language or **computable scenarios** written in a computable educational modelling language) and **concrete or contextualized scenarios** (and not **abstract scenarios**). Scenarios produced by teachers are always concrete, which means they correspond to a precise context.

The formalization and reification criteria have always the same value, while the others are variable criteria: they can vary during the scenario life or, for the same teacher, they can be different from a scenario to another. We recall these variable criteria in the following table.

Table 1: our use of Pernin's variable criteria to characterize scenarios.

Purpose	Scheduled	Descriptive	
Granularity	Session	Sequence	Structuration unit
Personalization	Generic	Adaptive	
Constraint	Constraint	Open	Adaptable

All these aspects must be taken into account in the descriptions. But a didactical study requires also additional features. We describe them in the next part.

METHODOLOGY

The data collected in 2005-2006 have two main origins: the five teachers of our group, and twenty other teachers, who followed a training course about a specific EEB called Mathenpoche ("Maths in the Pocket", shortened as MEP). For the sake of brevity, we do not mention here the questionnaire results. Moreover, we are now planning interviews useful to complement the questionnaires answers.

The teachers of our group are three primary school teachers (students between 9 and 11 years old); and two secondary school mathematics teachers (students between 12 and 15). Some are experienced with EEBs and some discover it, we explain it further in the next part. They were chosen to obtain this variety of contexts and experiences. During the whole 2005-2006 year all of them described precisely their uses of EEB, including: their preparation work, the description of what they planned, the description of what actually happened, the work done afterwards. Some of their sessions were observed, but only a few of them; thus the results presented in part 4 stem from the analysis of their own descriptions, formulated along a grid elaborated by the group.

The main resource used by the teachers of the group was MEP. We do not intend here to promote or dismiss any resource or kind of resource. MEP was the teachers' choice, thus a short insight of MEP's main features, and the presentation of the grid are necessary to understand our analysis.

MEP's main features

MEP offers exercises covering grade 6 to 9 curriculum, and a small part of grades 5 and 10. They are organised in sets of five or ten exercises with a common structure or theme; a given set of five exercises is identified by its title. Within an exercise set, the screen proposed to the student displays the text of one of the exercises (called a "problem", or a "question") with an answer zone to be filled in; the mark of the student (out of five or ten at the end of a given set of exercises); a button "calculator" providing access to a simple calculator; and a button "help" providing access to a full, explained solution of a similar exercise (always the same for a given exercise set). The expected answer can be numerical; some exercises offer multiple choices. After submitting their answer, the students receive a "Right" or "Wrong" feedback from the computer. Moreover, one or two detailed solutions of the exercise are displayed if their answer is right, or after a second wrong answer.

MEP is a free resource; some teachers use it with their classes without being registered as MEP's users. But registering as MEP's user opens more possibilities. The registered teachers inscribe their students into MEP. Each student is identified by a login and a password. Then the teacher chooses the exercise sets (these sets cannot be broken into smaller pieces) s/he wants to present to the students. The choice can be the same for the whole class, or different for subgroups, or even for individuals. The path of the students among the exercise sets selected can be left free or can be restricted according to the teacher's preferences: for example the second set could be offered only after a given threshold mark has been reached for the first. The registered teacher can follow the students' activity directly, through a special screen, during the sessions, or later by reading the "sessions' sheet" (see figure 1 below). It provides for each student (or group of students working on the same computer) the exercise sets tackled during the session, the time spent, the average, maximum and

minimum mark obtained, and for each exercise, the success (green or medium grey in a not coloured print) or failure (red/dark grey; the question not tackled are displayed in blue/light grey).

Figure 1 Extract of a session's sheet.

Exercices abordés.	abordé	moyenne	Mini	Maxi	Temps moyen
"Proportionnalité ou pas ?" (6N5s1ex3)	9 fois	2 / 5	0	5	08' 38"
"Combien ?" (6N5s0ex1)	14 fois	4 / 5	0	5	15' 47"
"Problèmes de comparaison" (6N5s0ex3)	12 fois	3 / 5	1	5	17' 37"

-----Résultats par élève :-----

Nirta

bf mn - moyenne : 6 / 10, minimum : 6 / 10, maximum : 6 / 10

- "Problèmes de comparaison" (6N5s0ex3)  3 / 5 (33 min. 38 s.)

bg rf - moyenne : 9 / 10, minimum : 8 / 10, maximum : 10 / 10

- "Combien ?" (6N5s0ex1)  5 / 5 (15 min. 15 s.)
- "Problèmes de comparaison" (6N5s0ex3)  4 / 5 (21 min. 03 s.)

bl es - moyenne : 4 / 10, minimum : 0 / 10, maximum : 10 / 10

- "Combien ?" (6N5s0ex1)  5 / 5 (07 min. 48 s.)
- "Problèmes de comparaison" (6N5s0ex3)  3 / 5 (23 min. 39 s.)
- "Proportionnalité ou pas ?" (6N5s1ex3)  0 / 5 (05 min. 33 s.)

Naturally, the teachers in our group are registered as MEP's users; it is the main resource they used.

A grid for the description of scenarios' features

The grid for the description of scenarios' features was developed with the teachers of the group. It includes Pernin and Lejeune's criteria, but also all the other features that appeared in the teachers' informal descriptions.

They systematically focus their descriptions on the mathematical knowledge to reach. The source of their reflections is the knowledge to be taught. The question of using an EEB and the way to do so comes next. It can lead teachers to build their scenarios either at the session or at the sequence granularity level. These two possibilities appear in the following grid.

Table 2: grid for the description of scenarios' features

1. Work plan	1.1 Resources: chosen EEB, other ICT, other supports
	1.2 For a sequence : sessions distribution and articulation, number, time
	1.3 Mathematical content, objectives
	1.4 Session(s) type(s): discovery, remedial, evaluation... ; precise the EEB function
	1.5 For a sequence: references to the EEB in paper and pencil sessions

2. Teacher's interventions during computer session(s)	2.1 Content
	2.2 Support
3. Students' activities	3.1 Written notes expected during computer session (s) and written documents provided
	3.2 Alone / pair / group work during computer session(s)
	3.3 Work with computer out of class
	3.4 <i>Differentiation (Personalization)</i>
	3.5 <i>Progression: imposed, with thresholds, free (Constraint)</i>

The grid comprises components stemming from our didactical approach and others corresponding to Pernin's variable criteria (italicized in the table). The teachers filled in such a grid for each of their uses of an EEB. We now present the main results, in terms of scenario's evolution observed.

GENESIS PROCESSES OBSERVED IN THE GROUP

We present in the following table basic numerical data about the use of web resources by the members of our group.

Table 3: Use of web resources during the year 2005-2006

	Teacher 1 (Secondary school)	Teacher 2 (Secondary school)	Teacher 3 (Primary school)	Teacher 4 (Primary school)	Teacher 5 (Primary school)
Number of scenarios	7	6	5	6	4
Use of the resource in class (hours)	15	10	15	9,5	8

These data help to figure out what happened in the classes; however, we are more interested in the evolutions observed. Two teachers in the group were familiar with the use of MEP, it was indeed their third year using MEP in their classes. They both used MEP, but also another EEB completely new for them. Two teachers already met MEP at the end of the preceding year (and used no other resource); we consider that they started being familiar with it around January, when they overcame all the

technical difficulties. One teacher discovered MEP by joining the group (she also used two other resources). According to the description of scenarios provided by the teachers, we retain several kinds of noticeable evolutions during the year 2005-2006.

From isolated sessions to sequences

Two main granularity levels appear in the teachers' descriptions: the session and the sequence. Less frequently, other time scales are mentioned, corresponding to specific uses: training out of class, programmed for several weeks or even months; or individual help. But the work intended for the whole class is thought of on the session, or on the sequence granularity.

The ability to plan at the sequence's scale appears as a result of the genesis process. For unfamiliar resources, 12 scenarios are proposed; 6 are planned for one-hour sessions, and 6 for sequences. For familiar resources, 16 scenarios are proposed; 13 for sequences and only 3 for one-hour sessions. The discussions in the group confirm these observations. With a new resource, the teachers have a wide mathematical content to discover, they can not rely on well-known exercises to build a progression. For example, when the two secondary school teachers discovered the resource WIMS (Web International Multipurpose Server, <http://wims.unice.fr>), they retained a ready-made session of training about powers of ten. Discovering all the exercises in this theme to build a progression was a too tough work (understanding the resource features already required two hours). Thus they only organized this isolated training session; it was related to their traditional teaching on the theme, but not really integrated in it.

On the opposite, a familiar resource is an instrument for the teacher just like the textbook for example. For a given mathematical content, the teacher plans a sequence, and foresees where the EEB will intervene if he or she already knows that the EEB proposes interesting exercises about this content. For example, for the theme "parallel and perpendicular straight lines" for grade 6 pupils, a sequence of 11 hours was planned, including 2 hours on MEP, 2 hours on a dynamic geometry software, and 7 paper and pencil hours. All these kinds of sessions were strongly intertwined; the aim of the work on MEP was described as "observing figures and discovering first proof principles", and a synthesis was organized in class after it.

A wide range of scenarios

The scenarios elaborated by the teachers of the group display an important variety of features; we emphasise here three main directions where evolutions were observed during the year.

About the function of the EEB: the teachers unfamiliar with an EEB mostly use it for training on technical abilities, like drill and practice software. Another use planned by unfamiliar teachers is the projection of the solving of an exercise with a video for the whole class. Both cases share a common feature. They avoid the discussion in class

about exercises done by the students on the computer. The teachers in the group unfamiliar with MEP expressed it several times: they feared to organise a discussion after the work on MEP, because the students met different exercises on the computer, and the teacher found difficult to discuss in class contents that some students perhaps had not met. This obstacle was overcome mostly thanks to the session's sheet, that allows the teacher to identify the exercises tackled by all the students. Then it became possible to use MEP to discover new techniques and properties; the session's sheet also made possible the use of MEP for evaluation purposes. The evaluation function does not appear in the scenarios organised by teachers unfamiliar with MEP.

About the written notes expected during the computer sessions: written notes seem to be expected more frequently with familiar resources. It is naturally connected with the preceding point: written notes do not seem so necessary during a drill and practice session, because there will be no further work on such a session. Moreover, it is easier to choose the appropriate written note for familiar exercises. It is indeed difficult for the students to write everything they do on the computer. Thus a choice is necessary. For example, at the end of the year, one teacher asked the students to write down the full solution of one MEP's exercise for each exercise set (five exercises) tackled.

About general organisation choices: the whole class can work at the same moment on the computer, students working by pairs. Or subgroups can work individually on the computer, while the other students have a paper and pencil work. Out of class work has also been planned by one primary school teacher. These choices are naturally strongly related with technical conditions. A half class on the computer, and the other half class on paper is only possible if the computers are situated in a room with enough additional tables. And the work out of class was only programmed by teachers who know their pupils have an Internet access at home, or at a nearby library. But the increasing confidence of the teacher with the EEB seems also crucial. A good knowledge of the exercises permits for example to retain exercises interesting for exchanges within a pair (for example exercises with two possible solutions).

The discussions with other teachers during the group meetings also played an important part in the development of various scenarios. The only teacher who was not able, for geographical reasons, to join the group's meetings always proposed similar scenarios (sequences of around ten sessions, with one or two hours on MEP for drill on technical exercises without written notes). On the opposite, the other teachers clearly influenced each other; two teachers working in the same secondary school regularly worked together to prepare their scenarios, and even ended up with a scenario associating two of their classes split in three level groups.

More personalization and constraints

The scenarios planned display more and more personalization and constraints. Two ways of personalization and constraints are used: one is the use of the possibilities

offered by the resource (mostly MEP); the other is the formulation of advice by the teacher.

Some teachers program a different content for different groups of students, even sometimes for individuals. This is the personalization explicitly intended by the designers of the resource. At the end of the year, three of the five teachers chose on the opposite to propose a wide range of exercises, the same for all students. The personalization was then done through individual advice on the exercises to tackle.

Similarly about constraints, some teachers use the constraints proposed by the resource: they program sessions where exercise 2 can only be tackled after exercise 1 (imposing a threshold mark was quickly let down). But at the end of the year, they seem to prefer proposing all the exercises together, and formulating their own constraints, the most frequent being about the number of attempts on the same exercise: never more than two attempts, whatever the mark is.

In both cases, the teachers first become acquainted with the possibilities offered by the resource; then they elaborate their own possibilities, sometimes not expected by the designers of the EEB. They create their own instrument from the EEB.

CONCLUSION AND PERSPECTIVES

We clearly observed evolutions of the teachers' practices. These evolutions are linked with their mastery of the technical features of the EEB, but mostly with their knowledge of its mathematical content.

We demonstrated that the practices of the group's novices teachers evolved towards the experts' ones. It allows us to consider the experts' practices as stable behaviours. The teachers of the group built an instrument from the EEBs they used.

It supports the interpretation of the practices' evolutions as teachers' instrumental geneses. Thus the next step of our work is to describe techniques considered as the visible part of the schemes developed by teachers. First, we have to identify marks of instrumentation and instrumentalization phenomena in teachers' practices with EEBs.

Tasks and techniques in the teacher's action have been studied by Sensevy and al. (2005). These authors identify tasks and techniques in the teachers' practices through direct observation and through the teachers' descriptions of their own practices. In a similar way, we are now trying to identify, in the scenarios described by the teachers but mostly through class observation, techniques instrumented by the EEB. For example, the EEB seems to provide instrumented techniques for the task "managing the students heterogeneity", because it permits for example 1. an instrumented diagnosis of students' knowledge by choosing appropriate exercises, 2. an instrumented choice of contents by programming individual menus and 3. an instrumented heterogeneity's management, allowing for example teachers to follow the students' individual work.

But studying the instrumented techniques requires an analysis focused on a particular mathematical content, allowing a comparison, on more precise tasks, of the teachers' paper and pencil techniques and their techniques with the EEB.

REFERENCES

- Cazes, C., Gueudet, G., Hersant, M. and Vandebrouck, F. (2005) Problem solving and web resources at tertiary level. *Actes du Colloque CERME 4, San Feliu de Guixols, Espagne.*
- Gueudet, G. (2006) *Learning mathematics in class with online resources* 17th ICMI study, Hanoi.
- Guin, D., Trouche, L. (eds.) (2002) *Calculatrices symboliques. Faire d'un outil un instrument du travail mathématique, un problème didactique.* Grenoble : La Pensée Sauvage.
- Koper, R. (2001) Modeling units of study from a pedagogical perspective. The pedagogical meta-model behind EML. Open university of the Netherlands.
- Monaghan, J. (2004) Teachers' activities in technology-based mathematics lessons', *International Journal of Computers for Mathematical Learning* 9, 327-357.
- Lejeune, A., Pernin, J-P. (2004) *A taxonomy for scenario-based engineering, Cognition and Exploratory Learning in Digital Age Proceedings*, 249-256, Lisboa, Portugal.
- Rabardel, P. (1995) *Les hommes et les technologies, approche cognitive des instruments contemporains.* Armand Colin, Paris.
- Sensevy, G., Mercier, A., Schubauer-Leoni, M-L., Ligozat, F. & Perrot, G. (2005) An attempt to model the teacher's action in mathematics, *Educational Studies in mathematics*, 59(1), 153-181.
- Trouche, L. (2004) Managing the complexity of the human/machine interaction in computerized learning environments: guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning* 9, 281-307, Kluwer academic publishers.

STRANGE 3D PLOTS¹

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The study of a real function of two real variables can be supported by visualization using a Computer Algebra System. One type of constraints of the system is due to the algorithms implemented, yielding continuous approximations of the given function by interpolation. This masks often discontinuities of the given function and can provide very strange plots. The pixelization adds to this strangeness. We present a study of such a "distortion of reality", using more than one system, and show how they can be used as a starting point to learn more Mathematics. This can be an obstacle, but we see it as a chance, as it is a motivating constraint, both for students and for educators.

I. GENERAL FRAME OF STUDY.

With the introduction of the computer into the learning environment the way Mathematics is conveyed has changed. The traditional sequence Definition-Theorem-Proof has received a complement with numerous examples, where visualization plays an important role. Various questions about the study of functions and curve discussion have been studied by Koepf (1995), Kidron and Dana-Picard (2006) and others. Many educators have replaced the traditional sequence mentioned above by another one, beginning with computer assisted experimentation.

The usage of the computer is not neutral: it brings with him a set of constraints. Some of them are limiting constraints, either due to the physical architecture of the device or due to the logical structure of the software and to the specific algorithms implemented in it. Following Balacheff (1994), Guin and Trouche (1999) distinguish three kinds of such constraints, namely *internal constraints* (of the hardware), *command constraints* (due to the existence and to the syntax of the commands), and *organization constraints* (linked to the interface between the artifact and the user). The need not only to receive the immediate result but to understand what is performed by the computer has revealed a fourth kind of constraints, called *motivating constraints* and analyzed by Dana-Picard (2006). These constraints

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provide a strong motivation to the student to learn more mathematics than what he/she knows presently (Artigue 1997, pages 139-140).

In every learning process, new notions are acquired and need to be internalized, routinized by the student and then they become de-mathematicized (Artigue 2002). The student does not ask himself/herself about mathematical meaning of the related actions anymore and the mathematical language has become "natural" language. In fact, for (Chevallard 1992a, 1992b) the mathematical objects have no real life per-se, but they emerge from practices. If they seem to exist per-se, this provides from this naturalization, and they become *transparent*. Of course, this is true for the mathematical usage of new artifacts, such as Computer Algebra Systems (CAS). The simple fact that millions of professionals and students have already used this technology gives confidence that most of the algorithms are correct and their answers accurate. This confidence is often too big. Problems appear when passing to the "next level" without questioning the validity of the old procedures in the new setting. The transparency can be exaggerated, and misconceptions can appear. The constraints of the artifact can lead to a strange situation, in particular when using the graphical features of the CAS. The display can be in conflict with the actual mathematical meaning of the situation; see (Giraldo and Carvalho 2003).

A well known case is curve discussion, where discontinuity of the function or the non-existence of a limit at some point are sometimes not shown by the display, despite the fact that a mathematical proof is easy to write. Such a cognitive conflict can appear, in a stronger fashion, when studying functions of two real variables. In order for the eye to catch the situation, most Computer Algebra Systems enable a dynamical point of view, using the mouse to turn around the surface. This can help to discover discontinuities or points where partial derivatives cannot exist, but how to be confident of the exactness of the visual impression? As we will see, discontinuities can be hidden.

Functions of two real variables are generally introduced in a Calculus II course, where the students discover generalizations of notions learnt in Calculus I. The respective roles of the first and second derivatives are extended in the new frame to discover extrema; saddle points generalize points of inflexion. When arriving to the visualization, drawings are harder to obtain by hand-work and computerized help is welcome. Plots are obtained via commands similar to the 2D-commands and the student is confident that what is seen is what *has to* be seen. The first problem is that, for example, a paraboloid drawn on the black/white board looks as in Figure 1(a), but generally a CAS plots something like (b) or (c), obtained with Maple.

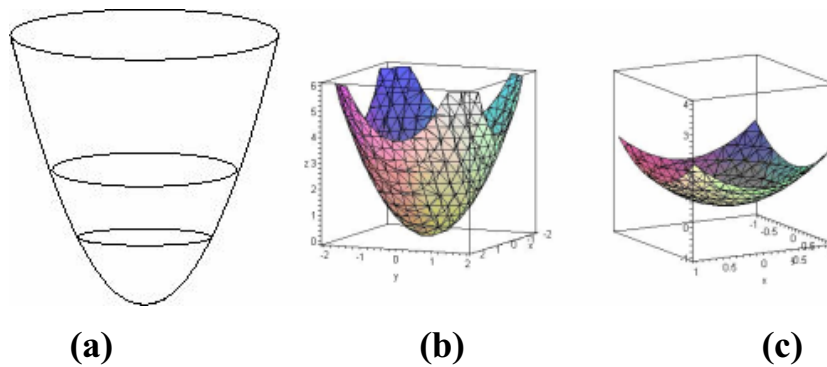


Figure 1: Looking at a paraboloid.

The reason for that is the fact that the CAS plots within a bounding box. The cube cuts the paraboloid in a visible way. In Figure 1(a), the general shape of the surface can be imagined, but this is doubtful for Figure 1(b). The different choices of the ranges for x and y are responsible for this disparity in the display.

Now consider the function defined by $f(x, y) = \frac{xy}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$, whose graph is displayed in Figure 2 for $-1 \leq x \leq 1, -1 \leq y \leq 1$ (obtained with DPGgraph: <http://www.dpgraph.com>). It "seems" that something happens in the neighborhood of $(0, 0)$, leading the student to check whether the function is differentiable at $(0, 0)$ or not. The grid itself sustains the intuition that when approaching the origin on different straight lines, something changes, whence the discontinuity of the function at $(0, 0)$. Note that DPGgraph enables the user to animate the plot, whence having the opportunity of viewing the graph from numerous points of view.

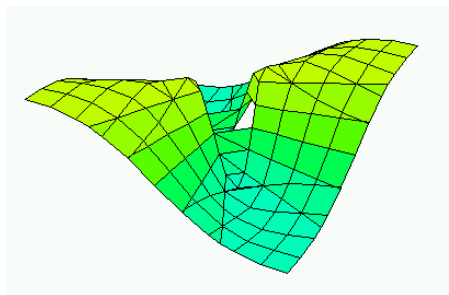


Figure 2: The function is not differentiable at the origin.

In this paper, we describe a much less comfortable situation. An algebra system is asked to plot the graph of a function of two real variables. Different systems, different commands of a CAS produce very different plots. And still worse, it happens that the plot does not fit the mathematical intuition. If we thought that both the mathematical practice and the technical skills which had been acquired by the student are really transparent, we discover now that they are not. We need to make these practices visible again; they must be the subject of a didactical analysis (Artigue 1997, page 138).

II. DIFFERENT PLOTS WITH DIFFERENT COMPUTER PACKAGES.

Consider the function of two real variables given by $f(x, y) = \frac{1}{1 - (x^2 + y^2)}$. It is a rational function, continuous all over the plane, excepted the unit circle U . We wish to study the behaviour of the function f near the unit circle. As the numerator is constant and the denominator tends to 0 when (x, y) gets arbitrary close to a point on U , the intuition says that the function has an infinite limit at every point of U . Moreover, as the denominator is negative out of U and positive inside, the cylinder given in the 3-space by the equation $x^2 + y^2 = 1$ plays the role of an asymptote to the graph of the function, the graph approaching the cylinder in two different ways, according to which part is consider, either inside or outside. Three different plots, obtained with DPGraph are displayed in Figure 3, according to the data:

(a) $-2 \leq x, y \leq 2, -3 \leq z \leq 3$, (b) $-3 \leq x, y, z \leq 3$, and (c) $-2 \leq x, y \leq 2, -5 \leq z \leq 5$.

The three plots are similar but show differences: the cylinder is always "nice", but the graph of the function presents various indentations:

- the inner-upper sheet being indented in (a), not in (b) and (c), but the z-values seem to become too large for them to be fully plotted in (c).
- the lower sheet has strong indentations in all cases, instead of being "full", as could have been expected.

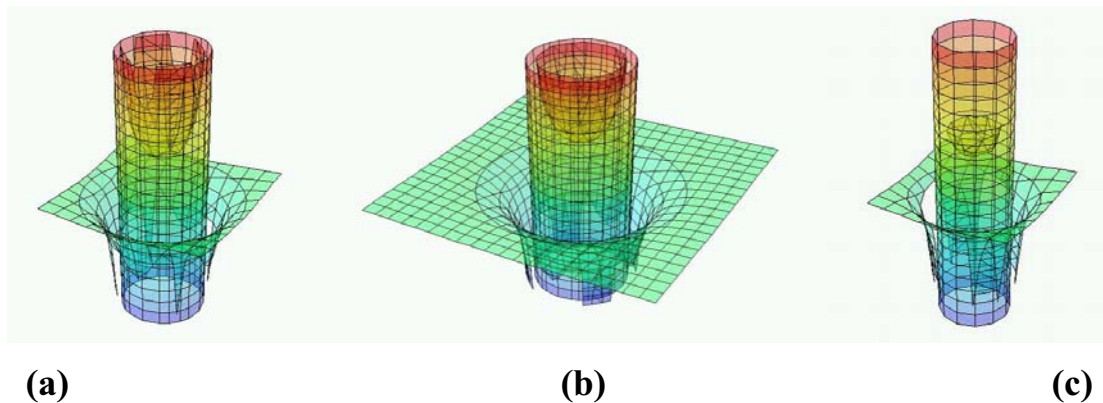


Figure 3: The graph and its asymptotic cylinder, using DPGraph.

The impossibility for a CAS to plot a graph close to a singularity has been already studied for plane curves, and various behaviours described by Koepf (1995), Dana-Picard (2005) and others. Here we have a similar situation for surfaces in 3-space, even more complicated as the singularities are not isolated.

Let us see what happens with another Computer Package. Figure 4 presents plots of the same function obtained with Maple's command `plot3d` ((a) and (b)) and

implicitplot3d ((c) and (d)), over different domains containing the unit circle. They look very different. What happened?

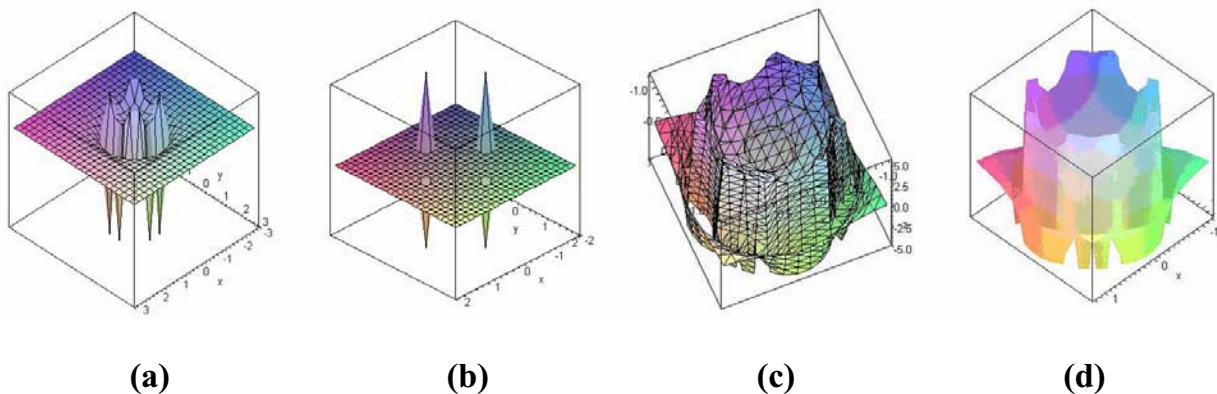


Figure 4: Maple plots with plot3d for various domains.

The various plots, obtained either with different commands over the same domain, or with the same command over various domains, look very different. Once again what happens? How can we use these plots in order to have an accurate graphical study of the given two-variable function? These questions lead to the study of the algorithms implemented in the different packages. They reveal different choices, inducing different epistemologies, as we will see in the next section. We wish to make clear that Figure 3 and Figure 4 have been obtained over the same domain respectively, only with different programs.

III. THE UNDERLYING MATHEMATICS.

We present it briefly; the interested reader can consult for example (Bradie 2006) and (Kahaner et al. 1989).

Graphic representation of two-dimensional real functions

In modern mathematical software such as Mathematica, Maple, Matlab and others, there are two types of algorithms for viewing mathematical functions:

1. Commands based on a Cartesian grid.

These three dimensional plot command (such as **plot3d** in Matlab) represent a real function of two real variables in a three dimensional view by approximating the function on a cartesian grid.

The two dimensional domain is *discretized* in a series of rectangular lagrangian elements, and on each element, the approximation is used. As an example, the two-dimensional Lagrangian interpolation arising out of linear bases defined on a rectangular element of size $h_x \times h_y$ are built as follows (we bring here the formulas for them to be *seen*, not necessarily understood now):

Denote $\xi = \frac{x}{h_x}; \eta = \frac{y}{h_y}$ Then $\phi_1(\xi, \eta) = [\frac{1}{2}(1-\xi)][\frac{1}{2}(1-\eta)]$, $\phi_2(\xi, \eta) = [\frac{1}{2}(1+\xi)][\frac{1}{2}(1-\eta)]$, $\phi_3(\xi, \eta) = [\frac{1}{2}(1+\xi)][\frac{1}{2}(1+\eta)]$, $\phi_4(\xi, \eta) = [\frac{1}{2}(1-\xi)][\frac{1}{2}(1+\eta)]$, etc. Lagrangian, quadric, cubic, and Hermitian cubic two – dimensional interpolation functions are presented by Lapidus and Pinder (1982), Section 2. These interpolations are often used in Computational Software.

In order to build a graph of a function using the computer, the domain is separated into a *finite number of elements based on* a collection of $n+1$ points $P_0(x_0, y_0), P_1(x_1, y_1), \dots, P_N(x_N, y_N)$, for each element: $[x_k, x_{k+1}] \times [y_j, y_{j+1}]$. On this element the function is approximated by a function built as above. For a given function $f(x, y)$, the interpolation used may be written as a double sum: a first summation over all the elements of the **discretization** of the domain, and a second one for the interpolation over the given element:

$$f_{app}(x, y) = \sum_{Elements} \sum_{i=1}^{i=4} f(x_i, y_i) \phi_i\left(\frac{x}{h_{ix}}, \frac{y}{h_{iy}}\right) \quad (*)$$

2. Command based on a parametric representation of the curve .

In this type of plotting the Cartesian coordinates are functions of two real parameters $x = x(s, t); y = y(s, t)$. A frequently used representation is the polar coordinates $x = s \cos(t); y = s \sin(t)$, where $s \geq 0, t \in \mathbb{R}$. Then the discretization of the domain is done on the parameters s and t .

In order to build a graph of a function using the computer, the domain is separated into a *finite number of elements based on* a collection of $n+1$ points $P_0(x(s_0, t_0), y(s_0, t_0)), P_1(x(s_1, t_1), y(s_1, t_1)), \dots, P_N(x(s_N, t_N), y(s_N, t_N))$, for each element: $[s_k, s_{k+1}] \times [t_j, t_{j+1}]$. On this element the function is approximated by a function built as above. For a given function $f(x, y)$, the interpolation used may be written as a double sum: a first summation over all the elements of the **discretization** of the domain, and a second one for the interpolation over the given element:

$$f_{app}(x, y) = \sum_{Elements} \sum_{i=1}^{i=4} f(x(s_i, t_i), y(s_i, t_i)) \phi_i\left(\frac{s}{h_{is}}, \frac{t}{h_{it}}\right) \quad (**)$$

The plot command yields a visualization as a surface plotting. The main difference between these two types of commands is in the mathematical treatment of the discontinuity and the critical points of the function.

Discontinuity and critical points of a two-dimensional function.

For a function $f(x, y)$ defined on a domain R , local maxima, local minima or saddle points can occur either at boundary points of R , or at interior points (x_0, y_0) of R

where the first partial derivatives vanish, i.e. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, or at points where f_x or f_y fail to exist. Here is the core of the strange apparitions in Figure 4: depending on the type of interpolation, the condition $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ is mathematically different from the condition $f_{app_x}(x_1, y_1) = f_{app_y}(x_1, y_1) = 0$ (***) .

For polynomial quadratic interpolation functions (Hermite and cubic basis functions), these last equations lead to a system of linear equations to solve in order to determine the (possible) local extrema. Such critical points exist on each discretization element of the domain. This simple analysis permits to understand why we are viewing plots with local extrema without any "connection" with the known mathematical behaviour of the function: **the local extrema plotted by the software correspond to extrema of the approximation function and not of the actually given function.**

For linear interpolation functions of cubic types (most frequently used in math software), it could be shown that the condition (***) is not dependent on the discretization steps h_{ix} and h_{iy} for a command based on a Cartesian grid (**plot3d**). However, the condition (***) for polar coordinates leads to a solution dependent on the grid discretization steps. Then, the viewing of local extrema on the graph based on Cartesian grid cannot be repaired by playing with the size of the mesh but may be repaired by playing with the discretization size and the domain range for plots based on parametric representations (**parametricplot**).

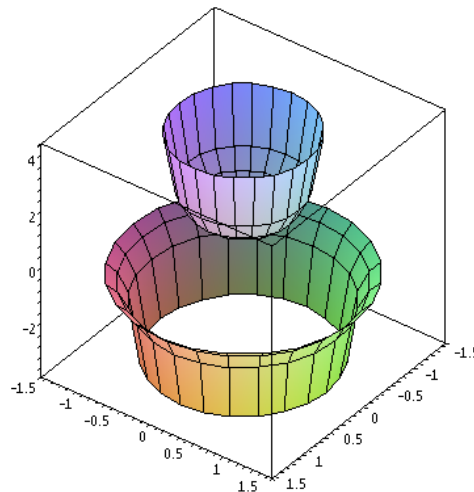


Figure 5: A plot in polar coordinates.

Transforming the question from cartesian coordinates into polar coordinates

An illustration of the problem may be seen for the different plots obtained for the given function. Figure 5 shows a surface plot of the above function constructed using polar coordinates ($x = r \cos t, y = r \sin t; r \geq 0, t \in \mathbb{R}$). In Figure 4, no discontinuity

appears near the circle whose equation is $x^2 + y^2 = 1$, but the plot in polar coordinates (Figure 5) shows clearly the discontinuity of the function.

IV. DISCUSSION.

When a student enters a course in multivariable Calculus, he/she is supposed to have achieved a certain level of routinization and internalization of the topics learnt for one-variable Calculus. As already mentioned, most notions in the new setting are generalizations of previous ones (limits, continuity, graph, etc.). This is true also for the contribution of the CAS to the exploration and the understanding of the mathematical situation (study of singular points of a function, etc.). Nevertheless, the passage from one variable to two variables creates new questions and new constraints. The educative gain of the described example has two components, of different importance, and different levels of difficulty. The first component concerns the teacher, the second one belongs to the student.

The first component is the need for the teacher to understand the internal structure of the computerized work (splines, approximations, etc.). This topic is quite hard for a beginner, and even teachers can be reluctant to enter it. Basic numerical analysis and the theory of approximation are generally not a part of the curriculum of a Teacher Training College. In an Engineering School, this takes 3 months in 3rd year. We do not expect to have first year students meet splines and approximations.

Concerning the second component, student oriented, we first ask 1) how to explain to students the different behavior of the CAS, 2) how the strange 3D plots can help foster the mathematical understanding of the students. Trying to answer the first question, the minimum that should be explained to the students is that he/she faces here one of the revelators of a double transfer. On the one hand, the given function has discontinuities, but at most points in its domain it is continuous. Even in a neighbourhood of a discontinuity of the given function, the standard approximation algorithms yield a continuous function on every element of the discretization of the domain. On the other hand, the display is made of pixels, i.e. it is discrete. We have here two sources for "a distortion of the reality". The second one has already been encountered, and analyzed, for plots of graphs of one-variable functions, and is not a novelty for a freshman in multivariable calculus.

Trying to answer the second question, we deal with the educative effect. When viewing the plots in Figure 4, the student faces a double conflict: a theoretical-computational conflict (TCC) and a computational-computational conflict (CCC).

- TCC: The computer seems to contradict the intuition of the existence of infinite limits at every point on the unit circle.
- CCC: For different domains, the plots look very different. Who is the right one, if there is "a right one"?

This confrontation with the strange 3D plots in cartesian coordinates (Figure 4) and the comparison with the plot obtained using polar coordinates (Figure 5) can help foster the students' mathematical understanding for the need to change coordinates. This need is not frequently met in the course. Maybe both conflicts TCC and CCC are resolved with this single process. This makes the mathematical proof of the infinite limit easier. As a consequence, instead of waiting until the need of polar coordinates to compute double integrals, the introduction of these coordinates should be made earlier. Here students feel the benefit of the Artigue's double reference (2002).

The two components give an example in undergraduate mathematics of the necessity of the double work requested by Artigue (1997, pages 139-140):

- The analytic work of identification of the constraints and the question of the epistemological validity of the software.
- The analysis of the functional and semiotic characteristics of the interface, and of the internal coherence and the tolerance of the device, which involves at the same time the internal world (the algorithms, the approximations) and the interface.

Dana-Picard (2005) claimed that it is possible to use technology in order to bypass the lack of mathematical knowledge of the student. This bypass provided an access to topics a little beyond the frontier of the taught syllabus, but not too far away. Here we are very far from a standard topic in a standard course in Mathematics. Therefore special attention has to be given to such situations, where neither the student nor (possibly) the teacher affords the required knowledge hidden in the algorithm. The general issue of choosing the appropriate CAS is irrelevant here, as every CAS plots surfaces in three dimensional space using approximations. The great originality here consists in the fact that we found a motivating constraint of the CAS which could *motivate the educator*, not only the student, to study more Mathematics than he/she used to. This claim has been verified by the colleagues of one of the authors, acting as the coordinator of the Calculus II course at JCT. We are quite far from the problems appearing within the processes of didactical transposition and computerized transposition; see (Artigue 1997).

An important question remains: should the implemented mathematical knowledge been always turned into a "teachable" mathematical knowledge? This question, already discussed by Artigue (1997, page 139) is beyond the scope of the present paper. Nevertheless, we wish to make the following remark, motivated by a question of the reactor. The problem that we present in this paper can be viewed as a severe obstacle to the understanding of the mathematical situation. In our eyes, it should rather be considered as a chance both to discover new mathematics and to make clearer the need to have a more critical attitude in front of the computer's results.

REFERENCES

- Artigue M. (1997), Le logiciel 'Derive' comme révélateur de phénomènes didactiques liés à l'utilisation d'environnements informatiques pour l'apprentissage, *Educational Studies in Mathematics* 33, 133-169.
- Artigue M. (2002). Learning Mathematics in a CAS Environment: The Genesis of a Reflection about Instrumentation and the Dialectics between Technical and Conceptual Work, *International Journal of Computers for Mathematical Learning* 7 (3), 245-274.
- Balacheff N. (1994). La transposition informatique: note sur un nouveau problème sur la didactique, in M. Artigue et al. (edts), *Vingt ans de didactique en France*, La Pensée Sauvage, Grenoble, 364-370.
- Bradie, B. (2006), *A Friendly Introduction to Numerical Analysis*, Prentice Hall, NJ: Upper Saddle River.
- Chevallard Y. (1992a), Concepts fondamentaux de la didactique: perspectives apportées par une approche anthropologique, *Recherches en Didactique des Mathématiques* 12 (1), 73-112.
- Chevallard Y. (1992b), Intégration et viabilité des objets informatiques dans l'enseignement des mathématiques, in B. Cornu (ed.), *L'ordinateur pour enseigner les Mathématiques*, 183-203, Nouvelle Encyclopedie Diderot, Presses Universitaires de France, Paris.
- Dana-Picard, Th. (2005): Technology as a bypass for a lack of theoretical knowledge, *International Journal of Technology in Mathematics Education* 11 (3), 101-109.
- Dana-Picard, Th. (2006), Motivating constraints of a pedagogy-embedded Computer Algebra System, to appear in *International Journal of Science and Mathematics Education*.
- Giraldo, V. and Carvalho, L.M. (2003): Local Straightness and Theoretical-Computational Conflicts: Computational Tools on the Development of the Concept Image of Derivative and Limit, *Proceedings of CERME 3*, available:
http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG9/TG9_Giraldo_cerme3.pdf
- Guin, D. and Trouche, T. (1999). The complex process of converting tools into mathematical instruments: the case of calculators, *International Journal of Computers for Mathematical Learning* 3, 195-227.
- Kidron I. and Dana-Picard Th. (2006), To see or not to see, *International Journal of Technology in Mathematics Education* 12 (3) , 115-124.
- Koepf W.(1995), Numeric Versus Symbolic Computation, *Plenary Lecture at the 2nd International Derive Conference*, Bonn, <http://www.zib.de/koepf/bonn.ps.Z>.
- Kahaner, D., Moler, C. and Nash, S. (1989), *Numerical Methods and Software*, Prentice Hall, NJ: Upper Saddle River.
- Lapidus L. and Pinder G.F (1982): *Numerical solution of partial differential equations in science and engineering*, John Wiley & Sons.

TOOL USE IN A TECHNOLOGY-RICH LEARNING ARRANGEMENT FOR THE CONCEPT OF FUNCTION

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Abstract

This paper describes the design of a technology-rich learning arrangement for the function concept. In line with the instrumental approach of tool use, aspects of conceptual understanding of the notion of function are linked with the tasks and with techniques in the technological tool, i.e. an applet embedded in an electronic learning environment. Some data from the pilot teaching experiment are presented.

INTRODUCTION

Since the origin of mankind, people have been using tools as extensions of the body. Vygotsky already pointed out that an instrument constitutes "a new intermediary element situated between the object and the psychic operation directed at it" and thus mediates the activity (Vygotsky, 1930/1985, p. 42). More recently, the complex nature of tool use has acquired considerable attention within cognitive ergonomics (Rabardel, 2002) and cultural-historical activity theory (Engeström et al., 1999). Research suggests a close relationship between tool use, cognitive development and social practice. Identification of the nature of these relationships and exploiting them for educational goals, however, is a non-trivial issue. The central problem with which education is confronted, therefore, is to identify the relation between the use of technological tools and learning, and to use these insights as guidelines for technology-rich teaching practices. How can the use of technological tools be embedded in innovative learning/teaching arrangements so that it improves learning?

A recent, promising technological development concerns small interactive environments called applets (or 'thinklets'), which are accessible through the Internet. For algebra, many applets are available, which aim at developing mental models and relating skills and concepts (e.g. see www.wisweb.nl). So far, however, neither the influence of the use of such applets in the teaching and learning of algebra nor the role of the teacher in such an arrangement have been systematically investigated.

This paper reports on the start of a research study on the use of applets for the acquisition of a rich function concept. The study focuses on students of grade 8 (13-14 years old), and is currently in its first phase. In line with an earlier pilot project (Boon & Drijvers, 2006) the aim is to investigate the power of combining the affordances of technology with by-hand work, group work and other forms of teaching.

RESEARCH QUESTIONS AND THEORETICAL FRAMEWORK

The three-year research project, entitled 'Tool Use in Innovative Learning Arrangements for Mathematics', started in January 2006 and focuses on the following research questions:

1. How can applets be integrated in an instructional sequence on the function concept, so that their use fosters learning?
2. How can teachers orchestrate tool use in the classroom community?

An important part of the theoretical framework of this study is based on the Vygotskian notions on tool use (Vygotsky, 1930/1985). Two related recent theoretical developments are relevant. First, the further development of cultural-historical activity theory by Engeström and his colleagues considers the activity system of which the tool, the user, the task and the social context are part (Engeström et al., 1999). The importance of the community of practice in which the learning takes place is stressed (Wenger, 1998). The notion of the activity system provides an interpretative framework to describe and analyse the social context of the classroom, and in particular interactions between students and guidance by the teacher. As such, it also addresses the second research question.

As a second recent elaboration of the Vygotskian notions on tool use, the instrumental approach to tool use stresses the distinction between the artefact – the object in use, for example a calculator – and the instrument (Lagrange, 2005; Rabardel, 2002; Trouche, 2004). The instrument consists of the artefact and the cognitive instrumentation schemes, which are needed to be able to use the artefact. In short, 'instrument = artefact + instrumentation scheme'. While constructing the instrument, students develop these mental schemes in which techniques for its use and insights into the concepts behind it co-evolve in a close relationship. By means of instrumental orchestration, the teacher guides this process of instrumental genesis (Drijvers & Trouche, in press). Because of the conceptual aspects within the cognitive instrumentation schemes, the instrumental approach is a valuable framework to investigate the relation between tool use and learning.

These two theoretical perspectives both stress the way tools mediate students' actions, as well as the relation between techniques for tool use, cognitive development and social context. The third component of the theoretical framework consists of domain-specific theories on the teaching and learning of mathematics. With the philosophy of realistic mathematics education as an overall background (Freudenthal, 1983), more specific theories on the learning of the function concept and the reification of function are considered (Vinner, 1983; Sfard, 1991).

RESEARCH SETUP AND METHODOLOGY

The research questions aim at 'understanding how' instead of 'knowing whether'. The research setup consists of three cycles of design research. Each of the cycles includes a design phase, a teaching experiment and a data-analysis phase. Throughout

the cycles, the length of the teaching experiment and the number of students involved increase. Also, the data that are gathered range from initially purely qualitative to more quantitative in the third cycle.

The design phase includes the design of student activities and tasks and the design of the technological tools that will be used. This is the main topic of this paper. However, the design of the teaching arrangements – the organization of the lessons and the way the teacher deals with theory, task and tool – is crucial as well. According to the instrumental approach, the orchestration of tool use is an important task for the teacher. By means of instrumental orchestration, the teacher fine-tunes the tool use in the classroom and thus, to continue the metaphor, conducts the different instruments formed by the students into one harmonic orchestra (Drijvers & Trouche, in press). In the design of the learning sequence, the issue of orchestration is dealt with by developing a teacher guide, in which didactical scenarios are described that suggest working arrangements as well as possible ways to use them.

During the teaching experiment, the following data are gathered. In the light of the first research question, video recordings are made of classroom teaching and group work; screen videos of pairs of students working with the computer are captured. Students' work with the applets is saved on a central server and photocopies of their written work are made. In the light of the second research question, the teacher's behaviour is videotaped, both in classroom teaching and in individual interaction with students. In this way, we hope to be able to trace exploitation modes of the didactic scenarios that are either more or less successful in the process of instrumentation and concept development. Data analysis is carried out by qualitative analysis and coding of the data; in the following teaching experiments, more quantitative techniques will be included.

This paper focuses on the design ideas concerning the notion of function, the role of the tools and the learning arrangement; it ends with some indicative data from the pilot teaching experiment.

FUNCTION CONCEPT

The study aims at developing an innovative learning arrangement for a rich and versatile function concept. This is not a trivial goal. Vinner and Dreyfus point out that there often is a considerable difference between the formal function definition and the students' concept image (Vinner, 1983; Vinner & Dreyfus, 1989; see also Meel, 1999). Malle (2000) describes two main faces of the function concept image: the input – output assignment and the co-variation, in which a change of the independent variable effects the dependent variable. The ideas of co-variation and dependency are stressed by Freudenthal as well:

The function is a special kind of dependence, that is, between variables which are distinguished as dependent and independent. (...) This - oldfashioned - definition stresses the phenomenologically important element: the directedness from something that varies freely to something that varies under constraint. (Freudenthal, 1983, p. 496).

Based on student interviews before the teaching experiment, we expect the starting function concept image for most of the students to be the local calculation procedure, which can be applied to one single input number at the time, and then results in an output number. The goals of the learning arrangement include the following four aspects of the function concept, which are of course not independent from each other.

1. An input – output assignment

The function is an input – output assignment that helps to organize and to carry out a calculation process. This somewhat vague notion gradually gets more nuances: *how* does the output depend on the input, *how* does the input determine the output? Its originally local character becomes more global: the function is not only a relationship between input and output numbers, but also one between domain set and co-domain. Functions can be compared with respect to global properties, such as increasing / decreasing or asymptotic behaviour. This opens the horizon for issues of co-variation.

2. A dynamic process of co-variation

This concerns the notion that the independent variable running through the domain set causes the dependent variable to run through the co-domain. The dependent variable co-varies with the independent. At first, the linked change is observed in a somewhat phenomenological way. Then, the question of *how* the process of joint dynamics is made to happen. What happens to the output if the input increases by 1 unit? How can one observe this in the table or the graph, or explain it with the formula?

3. A mathematical object

A function is a mathematical object which can be represented in different ways, such as arrows, tables, graphs, formulas, each of which shows a different 'face' of the same object. This concept image is an integrated function notion, which allows for reasoning with functions on a global level: how is a certain property of the graph reflected in the table or the formula, how can one decide if two functions belong to the same 'family'?

TOOL USE

In the learning arrangement, both material and digital tools are used. The material tools include a quadrilateral that can be changed (Fig. 1 left picture), posters to use in presentations, cards with operation symbols to make 'living operation chains' (Fig. 1 right picture), and cards with function representations to be matched. Typical in the design of the learning sequence is the aim to establish a close link between the material and the digital tools, by means of resemblance of representations. Here, we focus on the digital tools, i.e. an applet embedded in a simple electronic learning environment.



Figure 1 Sliding a quadrilateral and making a 'living operation chain'

The main digital tool that plays a central role in the learning arrangement is an applet called 'AlgebraArrows', developed by Peter Boon. AlgebraArrows (AA) offers means to construct input-output chains of operations. These chains can be applied to single numerical values as well as to variables. In the latter case, tables, formulas and (dot) graphs can be shown. Chains can be extended, linked, compared and compressed. Fig. 2 shows some of the main features of the applet.

The applet is embedded in a simple electronic learning environment, which is called the Digital Mathematics Environment (DME). The DME is an answer to the fleeting character of the work with applets, that both students and teachers often experience. In the DME, student work is saved on a central server. This allows the student to review the work, correct it, and continue it in any location with internet access, particularly at home. For the teacher, the DME offers a means of checking the students' progress, of monitoring the learning process by means of all-class result overviews and eventually of grading the students' work.

The DME also offers ways to set tasks and questions and for answering them outside the framework of the embedded applet. Fig. 2 shows how students can read the task in the upper left of the screen, work in the applet window on the right, and formulate their answer in a textbox on the left of the screen.

TOOL USE AND CONCEPT DEVELOPMENT

The instrumental approach to tool use stresses the relationship between developing a concept, using a technique with the tool, and carrying out a specific task (Kieran & Drijvers, 2006). How can the three aspects of the targeted rich function concept be connected to tasks in the learning arrangements and affordances of the tools, i.e. the applet in the electronic environment? Let us briefly go through the three aspects.

1. An input – output assignment

The chain of operations in the AA applet, applied to a single numerical value, reflects the function as an input – output assignment. In Fig. 3, two chains concerning different mobile phone offers invite a global investigation of properties. The technique of putting numerical values in the input window allows for comparison of function values. A variable input allows for a more global comparison, in which graphs, tables and formulas can be involved.

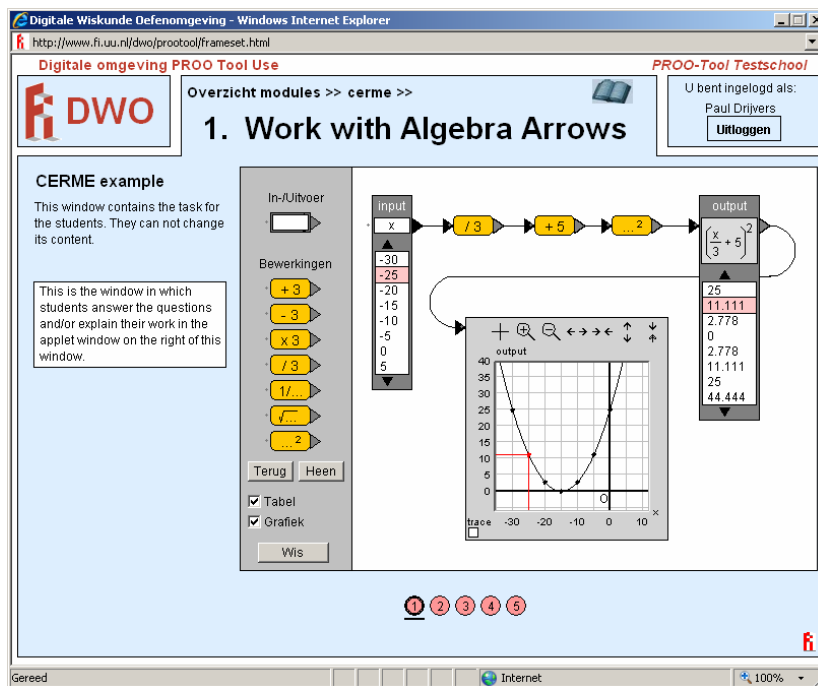


Figure 2 Main features of the applet AlgebraArrows embedded in the DME

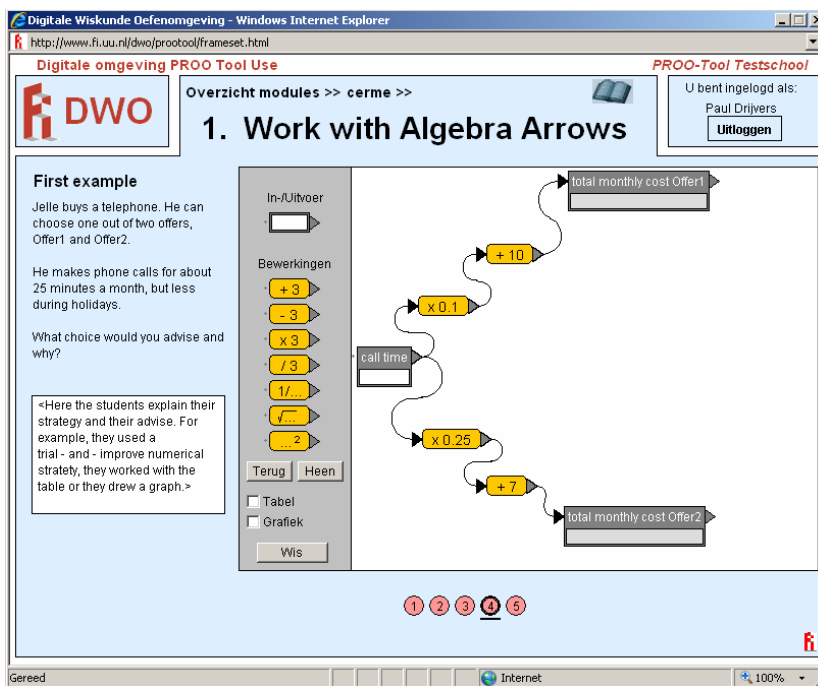


Figure 3 An input – output assignment

2. A dynamic process of co-variation

The applet allows not only a change of input value, but also the study of the co-variation by means of tables and graphs. The chain in Fig. 4 represents the cost of one particular mobile phone offer. The dynamics of the co-variation can be investigated by techniques of substituting different values, scrolling through the table, tracing the graph and studying the formula. The task for the students is to find the change of cost per minute change of the input variable – the number of minutes of phone calls.

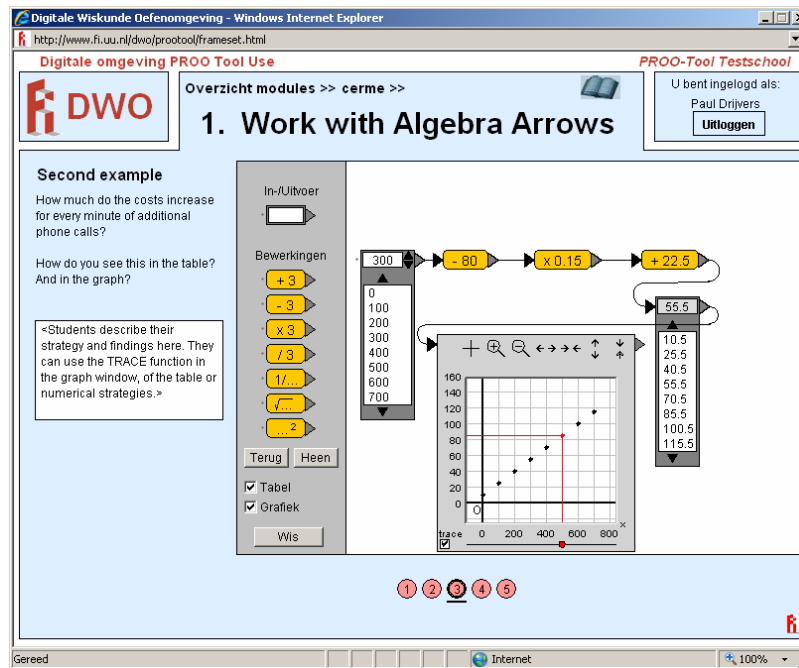


Figure 4 Co-variation

3. A mathematical object

The different representations available in the applet, combined with appropriate tasks focusing on the way they are interrelated, invite a more holistic view on a function as a mathematical object. A means to stress the object character of the function is to consider families of functions. Fig. 5 shows the chains and the graphs of the stopping distance as a function of speed for the case of a motorbike, a car and a truck. The technique of changing the last operation in the chain allows discovering the general features in the graphs and to find properties of the ‘family’ of functions, of which three members are represented.

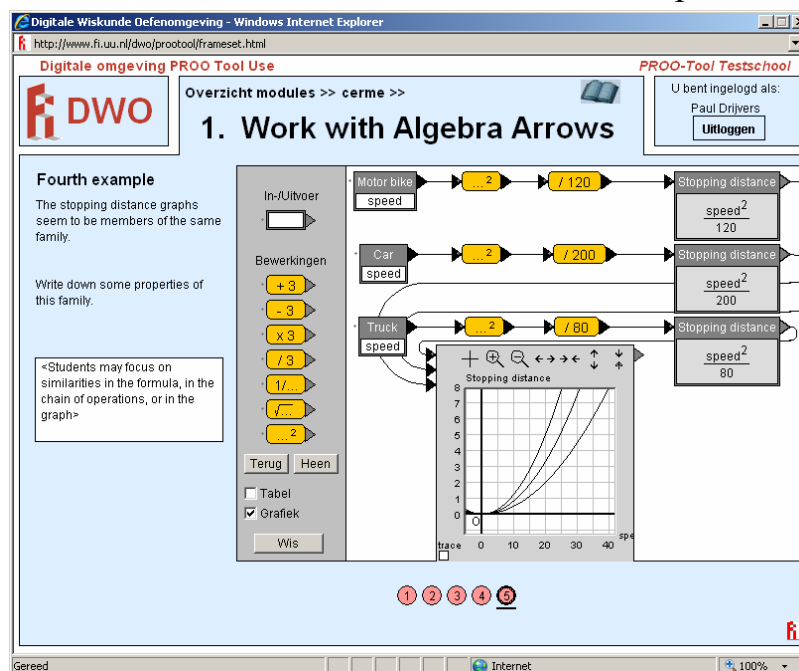


Figure 5 A family of mathematical objects

SOME FINDINGS FROM THE PILOT EXPERIMENT

Let us first briefly present some quotations from the student output windows.

- In the task presented in Fig. 3, one pair of students wrote down:
“He can better use Offer1 if he calls 25 minutes per month, but if he calls less than 20 minutes, he’d better choose Offer2”
Many students focus on the break-even point as the intersection point of the two graphs.
- In the task presented in Fig. 4, one pair of students wrote down:
“The numbers increase with 15 cents each time.”
An other pair was more vague:
“The minutes and the costs both increase.”
Many students noticed the co-variation, but had difficulties with clearly expressing their perception in a detailed way.
- In the task presented in Fig. 5, most students focus on the graph:
“They all increase in a kind of curve.”
In some exceptional cases, the chain of operations was considered:
“It gets squared in all three cases and then divided. But just divided by a different number.”
Once more, many students noticed similarities, but had difficulties to clearly express them.

Overall, it was noticed that the combination in the DME of applet facilities and ‘book facilities’ for posing questions and for answering them was powerful. Still, the quotations above indicate that students found it difficult to reflect on their work with the applet and write down precise mathematical conclusions. Also, the arrangement with strong links between the screen work and paper-and-pencil work had some practical complications. For example, once the students were working with the computer, they hardly looked at the hardcopy teaching materials. As a feed forward of the pilot, we decided to make the digital materials more self-containing through extra help facilities and digital copies of the written materials, to avoid the complexity of dealing with screen and paper at the same time.

The global learning trajectory turned out to work well, though it was noticed that it was too much driven by the mathematical concept and not enough by the problems that students could deal with by means of the applet. Also, we underestimated the students’ difficulties of choosing and naming the independent variable in applied situations.

The relation between applet techniques and conceptual thinking was clearly observed in the activities with the applet. However, it was not always clear whether instrumental genesis during the successive activities included student reasoning or was merely based on trial-and-improve. The options to enhance a dynamic view of function could be better exploited, which is one of the aims of the next research cycle.

In the post experiment interview, the teacher reported that the applet embedded in the DME was a powerful means to experience “that formulas, tables, graphs and all those things have to do with each other”. Also, she reports that at the end of the teaching sequence, students “did much better in reasoning than at the start”. However, she stated that technical classroom management was more complicated than in regular teaching, because of the mixed and integrated use of media and the somewhat complex teaching scenarios.

CONCLUSION

The pilot teaching experiment served as a first field test for the designed activities. Therefore, the conclusions mainly have a feed forward character for the next research cycle.

Concerning the first research question on the relation between the use of the applet and learning, we conclude that the applet offers interesting activities that foster conceptual development. The close relationship between applet techniques and paper-and-pencil activities guaranteed an integrated conceptual understanding and a transfer between applet notations and paper-and-pencil notations. In that sense, the ‘mixed media’ approach was fruitful. Still, a more detailed observation of the instrumental genesis process is needed. A more detailed description of the intended learning process will be developed, which delineates how successive techniques support each other and finally foster conceptual development. Such a description would allow to investigate whether the instrumental genesis is the result of learning history and imagery of the students or of trial-and-improve strategies (Gravemeijer et al., 2003; Doorman, 2005).

Concerning the second research question on the orchestration by the teacher, we conclude that the teaching arrangement was demanding for the teacher, who spent too much of her time on organizational matters. Furthermore, the DPME offered interesting possibilities for the teacher to monitor the students’ progress and to adapt her lessons to that. As far as teacher – student interactions during applet work is concerned, this turned out to be particularly helpful if the teacher’s explanation related to both technical and conceptual elements. Finally, applet work in pairs was not enough for collective instrumental genesis; classroom demonstrations and discussions played an essential role in capturing the students’ results and in provoking a collective convergence of mathematical thinking.

REFERENCES

- Boon, P. & Drijvers, P.: 2006, ‘Chaining operations to get insight in expressions and functions’. In Bosch, M (Ed.), *Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education*, Sant Feliu de Guíxols, Spain, 17 - 21 February 2005, <http://ermeweb.free.fr/CERME4/>, pp. 969-978.
- Doorman, L.M.: 2005, *Modelling motion: from trace graphs to instantaneous change*. Utrecht, the Netherlands: CD-Beta Press.

- Drijvers, P. and Trouche, L.: in press, 'From artifacts to instruments, a theoretical framework behind the orchestra metaphor', in M.K. Heid and G.W. Blume (Eds.), *Research on Technology and the Teaching and Learning of Mathematics: Syntheses, Cases, and Perspectives*, Information Age Publishing.
- Engeström, Y., Miettinen, R. & Raija-Leena, P.: 1999, *Perspectives on activity theory*, Cambridge University Press.
- Freudenthal, H.: 1983, *Didactical phenomenology of mathematical structures*, Reidel.
- Gravemeijer, K.P.E., Bowers, J. & Stephan, M.L.: 2003, A hypothetical learning trajectory on measurement and flexible arithmetic. *Journal for research in mathematics education*, 12, 51-66.
- Kieran, C. & Drijvers, P.: 2006, 'Learning about equivalence, equality, and equation in a CAS environment: the interaction of machine techniques, paper-and-pencil techniques, and theorizing' *International Journal of Computers for Mathematical Learning*. <http://www.springerlink.com/content/u7t3580294652u37/>
- Lagrange, J.-b.: 2005, 'Curriculum, classroom practices, and tool design in the learning of functions through technology-aided experimental approaches', *International Journal of Computers for Mathematical Learning* 10, 143-189.
- Malle, G.: 2000, 'Zwei Aspekte von Funktionen: Zuordnung und Kovariation', *Mathematik lehren* 103, 8-11.
- Meel, D.E.: 1999, 'Prospective Teachers' Understandings: Function and Composite function', *Issues in the Undergraduate Preparation of School Teachers: The Journal*.
- Rabardel, P.: 2002, 'People and technology - a cognitive approach to contemporary instruments', <http://ergoserv.psy.univ-paris8.fr>.
- Sfard, A.: 1991, 'On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin', *Educational Studies in Mathematics* 22, 1-36.
- Trouche, L.: 2004, 'Managing complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations', *International Journal of Computers for Mathematical Learning* 9, 281-307.
- Vinner, S.: 1983 'Concept definition, concept image and the notion of function', *International Journal of Mathematics in Science and Technology Education* 14, 293-305.
- Vinner, S. and Dreyfus, T.: 1989, 'Images and definitions for the concept of function', *Journal for Research in Mathematics Education* 20, 356-366.
- Vygotsky, L.S.: 1930/1985, 'La méthode instrumentale en psychologie', in B. Schneuwly and J.P. Bronckart (Eds.), *Vygotsky aujourd'hui*, Delachaux et Niestlé, pp. 39-47.
- Wenger, E.: 1998, *Communities of practice*, Cambridge University Press.

ANALYSIS OF TEACHER EDUCATION IN MATHEMATICS AND ICT

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Abstract: this contribution draws from a PhD work “Engineering teacher’s training about Mathematics and ICT”. We tried first to test the functionality of theoretical approach of teacher’s activity, in order to analyse teachers’ education courses. Our analyses are based on teacher’s educator interview and on observation of three teachers’ education courses. Knowing more about teacher’s education allows us to make hypotheses to build teachers professional development scenarios.

COMMENT ABOUT VOCABULARY

As far as teacher education is concerned, the French vocabulary cannot be directly translated into English. In French a pre on in service teacher attending a training or professional development session is named a “stagiaire” and the teacher or lecturer doing the sessions is a “formateur”[1]. A “formateur” is both a teacher trainer in the sense that he/she helps teachers to develop teaching skills and a teacher educator in the sense that he/she helps teachers to have a conceptual view of their activity. In this article we translate “formateur” into “teacher educator” and “stagiaire” into “trainee”.

QUESTIONS ABOUT TEACHERS’ EDUCATION IN MATHS AND ICT?

Our work wants to contribute to explain why it is so difficult for teachers to change their way of teaching in order to introduce ICT into mathematics teaching. We choose to focus on teachers’ professional development courses in Maths and ICT because we conjectured that these courses should help teachers to perform this introduction, but, in their present state, generally fail. Our objective is to analyse teacher educator practices in these courses and their influence on teacher’s activity, beliefs and attitudes.

We had then to specify a theoretical framework wide enough to help us observe and analyse teacher educators’ practices taking into account the trainees’ activity during the sessions and the influence of the session upon their practices in their classroom. This framework has to consider teaching activity at two levels. At a first level, teacher education is a teaching activity: the educator is teaching to a class of trainee teachers and at a second level, it is “teaching about teaching”, that is to say that the classroom situation –a teacher and his/her students- is the topic of the course. We start from an approach of teaching activity not specific to technology that we will use at the two levels. Then we consider an approach of the relationship between human beings and technology, specifying this approach to the situation of the teacher at the two levels. Finally, we go back to the issue of “depersonalisation” that was raised in a previous work about teacher development in ICT.

An approach to teachers' practices

Robert and Rogalski's (2002) twofold approach uses both a didactical and an ergonomic analysis of teachers' activity. Lagrange (2004) provides a short analysis of this approach: "The first component (didactical) takes into account the links between students' activities and the teacher's management. The second component (ergonomic) considers the teacher as a professional who is performing a specific job. Articulating these two approaches made it possible to see teachers' practices as a complex and coherent system, resulting from a combination of individuals' personal and professional history, knowledge and beliefs about mathematics and teaching".

Practically Robert and Rogalski offer to look at a classroom session through the "cognitive routes" that a teacher organises for his/her students. They distinguish five components, which can be observed or questioned:

- Personal component brings together teacher's representation of mathematics, teaching and learning process, students' needs and real life experience...
- Social component is the conditioning of the teachers' working environment, what is usual and unusual in the school where the teacher is working...
- Institutional component concerns the influence of institution (scholar curriculum, department of education or school inspector's directives). This influence can be explicit or not.
- Mediative component is related to all the interaction which could appear during the lesson, verbal or not.
- Cognitive component is linked with the knowledge which is at stake in the lesson, the mathematic content.

Each of these five components can be seen at the two levels considered above. For instance, we can observe the personal component of the teacher educator as an important factor in the training session, but also the personal components of the trainees and their evolution.

This is how we will put into practice this approach, considering our two levels. We will look at a training sessions through an analysis of the trainees' "cognitive route" organised by the educator. We will see how these "cognitive route" are interlinked with students' "cognitive routes" suggested in different manner by the teacher educator when presenting example of classroom use of ICT.

The instrumental approach

The instrumental approach developed by Rabardel (1999) allows us to take into account the specificity of ICT. The central idea is the instrumental genesis, by which a human being builds his/her instrumental schemes starting from an artefact. We look at the contribution of a training session on the instrumental genesis of the trainees at the two levels defined above. At the first level, we consider how the session contributes to the trainees' genesis of ICT tools. At the second level of analysis we

consider how the session helps trainees to develop instrumental schemes that could allow him/her to develop classroom uses of ICT. We conjecture that schemes that allow a teacher to use a ICT tool for him(her)self are not sufficient for effective classroom uses. Certainly, these uses demand a different knowledge of these tools, taking into account the classroom constraints and the development of students' relationship with these tools.

Personalisation and depersonalisation

In her PhD, Abboud-Blanchard (1994) investigated the body of knowledge that teacher educators aim to transmit during training courses. She brought evidence that this knowledge was "personalised" that is to say that the educator based his/her course on his/her own experience and views of teaching and of ICT. This "personalisation" made the sessions comfortable both for the educator and the trainees, but it was also analysed as a strong obstacle: it provided no means for the trainees to adapt the ICT use situations presented by the educator to their own professional experience and needs.

It is certainly not easy to conceive a sufficient "depersonalisation". It is a case where the interaction of the two levels defined above has to be carefully taken into consideration, because at the first level of the training session, the second level of trainees' professional situation and classroom activity has to be clearly at stake, in order to go beyond a mere evocation to the educator's experience.

Method and data

Little is known regarding teacher educators representations about mathematics and ICT, and about training in relationship with their representation of mathematic activity and teaching-learning processes. That is why the first dimension of our work was an analysis of these representations by way of interviews. We built up a questionnaire as a guide for an interview with fourteen Teacher educators [4]. The information we obtained helped us to analyse the educator's social and personal components as defined by the twofold approach. The questionnaire was based on results about teachers' representation of ICT, particularly those obtained by Rhéaume and Laferrière (2002).

In order to consider possible "cognitive routes" in training sessions, a second dimension was then to observe a panel of actual teachers' education sessions. We analysed three teachers' development courses ranging from three hour to twelve hour, concerning pre-service and in-service teachers in primary as well as secondary education. We were looking for regularities in training scenarios, in mediative component ...whatever the conditions may have been.

Our method was first to interview the educator before the course, asking him/her to explain his/her goals, what he/she wants to do, if he/she could predict the trainees' reactions ... then, to observe the session and finally after the course to ask him/her how he/she analysed the differences between what he/she had forecast and what

really happened. We also distinguished episodes in each session. This unit of analyse allow us to compare courses with different length. Interviews and courses have been recorded and transcribed.

RESULTS: AN ANALYSIS OF TEACHER EDUCATORS REPRESENTATIONS

Two types of teachers educators

Rhéaume and Laferrière (2002) defined categories of teachers with regard to ICT. From this work and our observations, we defined two large categories of teacher educators: “talented addicts” and “optimistic hard-workers”. This categorization is suitable for an analysis in term on instrumental genesis. The former have been using computers for a long time, and are using it for personal pleasure and self-interest. They are using ICT in teachers’ development by choice without any effort. These educators need no raisons to use an artefact, intellectual curiosity and pleasure to analyse this tool suffice. They see technical difficulties as a challenge that they are happy to accept.

The latter recently came to use computers, and mainly, at first for a utilitarian purpose. They decided to include ICT in teacher development because changes in curricula made it compulsory. When they encounter technical difficulties, they do not hesitate to ask for help. Didactical reasons drive their use of ICT, rather than a liking for technology.

Teacher educators’ views of ICT

We asked teacher educators their views about a variety of representations of teachers’ and learners’ activity, and the role of ICT in this activity. They generally agree about the role of IT for learning and in teacher Education. They reject intrinsic effectiveness of ICT for learning at the level of the classroom, while they think that ICT is essential for teacher education. The teacher is seen as being subjected to the irremediable development of technology and teacher educators have to help them to update their knowledge in order to be able to master their educational choices. Teacher educators also think that ICT is an efficient tool to individualize teaching.

In contrast, they do not agree about the effectiveness of Email or of collaborative platforms. We think that there is no consensus because web-based tools are recent: teacher educators' experiences regarding the use of web-based communication are dissimilar contrary to those of software application.

All teacher educators, excepting those working in one region, agree with the idea that students and teachers (especially pre-service teachers) can use ICT to cheat: for instance pre service teachers would cheat when they use the Internet to find classroom activities on decimal numbers instead of creating their own. It seems that local consensus can exist regarding social aspects of ICT, such as cheating.

Our conclusion is that teacher educators share representations about tools that exist since a sufficiently long time. These “old tools” and their representations by educators are part of a common professional environment. Educators’ views of “new tools” are more influenced by their personal experience and instrumental genesis.

Reasons for using ICT

We wanted to know the reasons why teacher educators are using ICT in different contexts. They give dissimilar reasons for using ICT with their students and in teacher education. Regarding classroom use with their students they say that students liking to use computers influenced them. They explain that they use ICT in teacher education because of curricular changes and institutional pressure. They are nevertheless vague about this institutional pressure and unable to quote precise curricular demands regarding ICT use. Our conclusion is that teacher educators consider institutional pressure as a general motivation for teacher education, but are more influenced by the way their students consider ICT.

Type of courses

For this analysis, we use the above idea of (de)personalisation. Abboud Blanchard (1998) distinguished three types of courses, from the more to the less personalized:

Type 1: the educator outlines an example of a lesson;

Type 2: the educator explains how the lesson has been built;

Type 3: the educator builds or rebuilds the lesson with the trainees;

In the interviews, teacher educators described courses that we could not classify into one of these three types. We added two types: one between the first and the second types where the teacher educator make trainees do the lesson like students would do, and a type after the third one: the teachers’ educator asks teachers to produce documents for dissemination (on institutional websites for instance).

Approaches of teacher education courses

We found two main approaches:

1. Starting from a piece of software (dynamic geometry or spreadsheet for instance), learning to use it, and then reflecting on how it could be used for learning;
2. Starting a mathematical concept and looking for a relevant contribution of ICT to teach/learn this concept (for instance, how to teach parallelograms categorisation and properties, with what software).

Talented addicts consider more the first approach and optimistic hard-workers consider rather the second. It is evidence that approaches to teacher education courses are influenced by educators’ own instrumental genesis.

Moreover, very generally too teacher education courses are not based on theoretical ideas. This lack of theory is consistent with the generally strong personalisation of

courses. It is also consistent with the type of evaluation generally adopted, simply based on teacher satisfaction.

What is a good teacher education course?

It was one of the questions we wanted to ask teachers' educators. In order to avoid stereotyped answers we used a trick: We reversed the questions and actually asked teacher educators to describe features that will most probably make a training session fail. We listed the answers from the most probable fail to the least and reversed the answers. We then obtained this list of features ordered from the most to the least important for a good training in the teachers' educators' view:

1. In a good session, the teacher can use the content immediately in their classrooms.
2. In a good session, contents are understandable.
3. A good session is when teachers feel good and work in good conditions.

Teacher educators also think that teachers are mainly expecting them to tell how to teach with ICT. They also list the main obstacles to the use of ICT: ICT equipment is poor in schools and usual teachers' way of teaching has to be changed.

Conclusions about teacher educators' representations

What teacher educators say shows us that there are common representations we can use in our analysis described below. Teacher educators' instrumental genesis influence their approach of teachers' development is concerned.

All these data also give us information about what a teachers' educator could accept to do. This led us to make hypothesis of viability of teachers' education sessions.

ANALYSIS OF ACTUALTEACHERS' EDUCATION COURSES

In this section, we use the above theoretical frameworks to analyse three complete teacher development sessions. In the first two parts we analysed the "level" of the teachers' educator. When we analyse the situations exposed during the course we notice what in term of the twofold approach that few dimensions of the practices are addressed. In the two last parts we analyse the level of the teacher using examples of a courses.

The structure of the sessions

We considered episodes, that is to say units of content and of task for the trainee. Every episode had the same structure: first, the educator presents a classroom situation, and then he asks trainees to behave like students and make them experience the situation. The episode finishes by a debate about technological and educational issues.

We noticed that during more than fifty percent of the time trainees are working independently. During the rest of the time, between ten and twenty percent is

dedicated to a discussion. This means that educators are speaking to trainees about eighty percent of the time without attending any answers. We noticed the lack of feedback. During courses, the most important part is dedicated to explanations and descriptions.

The type of the three sessions is between the first and the second of the above classification: the teacher educator makes trainees do the lesson like students would do. When we asked teacher educators about their goals and their approach they appeared to be aware of the structure and type we have brought to light. There is some inconstancy between the representations of the educators analysed above and what they do in their training courses: they can evoke depersonalised types of session that they do not actually put into action; they recognize the importance of reflection on ICT use, and in their actual sessions they let teachers in front of computer most of the time.

Types of classroom's organisation pointed up

Two material modalities are exposed: using a classroom devoted to the use of computers or being in a standard classroom with one computer linked to a PC projector. Two educational modalities are outlined: using computers to stimulate discussions between two students working on the same computer and making backwards and forwards between paper-pencil and computer work.

In those points we can find the results of analysis of teachers' educators. ICT is not a goal but a tool so it must be put at a distance. Furthermore teacher educators see teachers' practices as an obstacle to the use of ICT, thus they try to expose elementary material modalities.

How the course takes social, personal and institutional components into account

In this example of teacher development course dealing with the use of different tools to teach the parallelogram with 7th grade students we noticed that social, and institutional components are not taken into account. This need is underlined at the beginning of the session by several trainees' take of floor like:

Trainee: Next year, we will have groups in 7th grade..." [...]

Trainee: And you work with all the students?" [...]

Trainee: And you work with them during the whole hour in front of computers?" [...]

Trainee: In our school, the local education authority asked us if we wanted to receive hardware" [...]

First sentence show that teachers need to explain the context of their work. They also want to have information about uses in others schools, if the teacher goes in computer classroom during a whole hour or if he can makes different groups. All the answers given are personal, that is to say that teachers' educator explain his own practices without any distance. In the three teachers development courses studied it's the only

place where teachers' professional background is explicitly taken into account but in a loose talk.

Teacher educators are guided by the wish to give teachers ideas of activity suitable for them. In the frame of this matter, underlined in teachers' educator representation, they are taken into account those components but without verifying that what they call "suitable" is suitable for those teachers. In fact social and personal component are imagined by teachers educator.

Institutional component is never addressed. Curricula or certificates are not quoted during the session and teachers do not ask any question about this.

Second level: Analysis of situations

In only one course we could see an analysis of the mediative component, by way of a video of classroom activities. In this course, teachers' educators choose to record only the work of two students in front of a computer so only interactions between students and computer were at stake. The analysis of the lesson carried out during the session focused on student activity in front of the computer. Interactions between teacher and student were not taken into account. Teacher educators gave few pedagogical explanations to introduce the analysis of hardware about parallelograms:

Educator 1 [8]: You have got the lesson; John (Educator 2) started by some summary. "All you want to know about the parallelogram?" And at any time, there are applets on a PC projector, students have seen these applets and when they make an exercise they have at any time a summary of this lesson ...

Educator 2: it was on Monday at the back-to-school time so I preferred to make a little summary of lesson after...

Educator 1: Yes you were careful.

The analysis of another three-hour session about spreadsheet for pre-service students gives us information about mediative and cognitive component at stake. Mathematical concepts were quite never quoted. The first situation dealt with black boxes [7]. They were used by teachers' educator to link cells and variable in pre algebra. Functions used in black boxes are not analysed or questioned. It is only a pretext to study this use of a spreadsheet. The second situation deals with statistics. Spreadsheet is used to compile data entered by different students in order to make a concrete approach of statistics thinking.

Instrumental needs for such situations are weak and not taken into account during the course. First student must be sure that spreadsheet has a logical and rational behaviour. In the case of black boxes it is a support for the teacher to present a problem rather than a tool for the student. In the case of statistics, students had only to enter data.

In mediative component, responses of students and teacher are only related like in this quote:

Educator: So you choose an empty cell, how to calculate $2+3$? Students suggest to type what they would have typed in their calculator « $2+3$ [enter] » and they realize that it does not work. [...] then, usually students do not find the good syntax by themselves. For real it is not logical. I try to help them, I say: «You must advise the computer that it is neither a number nor a text so you have to put a symbol first» [...] so they type “ $=2+3$ [enter]” and then they obtain five. Further, lets go, it is the first session, it last a full hour [...].

There is quite no analysis rather a mere description.

Conclusions about teachers' development courses

At the first level of analysis we noticed regularities in courses structure and contents. A great part of the session is devoted to the use of computer. It seems to be an inherent but necessary condition. Teachers' educators aim to convince trainees by explaining them what can be done. The approach seems to be linked with the teachers' educator personal instrumental genesis; it is a form of personalization.

At the second level of analysis we noticed lacks: few aspects of teacher's activity are treated. Except some pedagogical points and some descriptions student are away from the courses. Personal, social and institutional are not taken into account but appears in the sessions in the form of not formal questions of trainees.

PERSPECTIVES

The analysis exposed above form a part of a current work about a little studied point. We are aware of the lake of coherence between the theoretical frameworks and the results. This aspect will be treated in our PhD. This analysis also helps us to make a hypothesis on what could be teachers' development helping them to change their teaching practices and to use ICT. The use of video seems to be good solution. We have to find a suitable methodology to test our hypothesis.

Researchers considered the idea of didactic engineering [9] for a long time: they design teaching courses to test hypotheses about students' learning. We decided to also use this approach to test our hypothesis about teacher's development, after adapting this approach to teacher education.

Specific difficulties

In the case of teaching to student it is rather simple to verify these hypotheses: teaching courses can be tried out with different groups of student. Teacher development courses last only a few hours and their effects are very difficult to isolate from other causes. We will have to choose a suitable manner to validate our hypothesis.

NOTES

1. Translated word by word in « former » or « teachers' educator ».

2. Translated word by word in “probationner” or “intern”.
3. Instituts Universitaires de formation des Maîtres: university institute for teachers training
4. Seven came from the Champagne Ardenne region and the others came from other parts of France
5. C2i : Certificat Informatique et Internet = certificate of computing and Internet for teacher; B2i for students, using spreadsheet software to programme Euclid’s algorithm is noted in secondary school mathematics syllabus ...
6. twenty two for primary education student teachers and twenty one for secondary education student teachers
- 7: black boxes are cells where student can see the result of a function without seeing the function itself. They can test many values and so guess how the function has been defined.
- 8: Charles and John are teachers’ educators in this session.
9. The didactic engineering concept has been defined by Y. Chevalard as a methodology of research cf. ARTIGUE in references.

REFERENCES

- Abboud Blanchard, M.: 1994, *L'intégration de l'outil informatique à l'enseignement secondaire : symptômes d'un malaise*, Thèse de doctorat, Université Paris VII, Paris.
- Artigue M., 2002, *Ingénierie didactique : quel rôle dans la recherche en didactique aujourd'hui*, les dossiers des sciences de l'éducation
- Lagrange J.-B., 2004, *Teachers using technology: models of the complexity of practices*, CERME 4
- P. Rabardel, 1999, *éléments pour une approche instrumentale en didactique des mathématiques*, conférence, Actes de l'université d'été, Université de Caen,
- Rhéaume J., Laferrière T., 2002, *Les communautés virtuelles d'apprentissage*, dans *Enseigner les TICE*, dirigé par R. Guir, De Boeck.
- Robert A., 1999, *Recherches didactiques sur la formation professionnelle des enseignants de mathématiques du second degré et leurs pratiques en classe*, didaskalia 15.
- Robert, A. and Rogalski, J.: 2002, ‘Le système complexe et cohérent des pratiques des enseignants de mathématiques : une double approche’, *Revue canadienne de l'enseignement des sciences, des mathématiques et de la technologie*, 2 (4), 505-528.

DEVELOPING TASKS AND TEACHING WITH ICT IN MATHEMATICS IN AN INQUIRY COMMUNITY

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In this article I report from a project aimed at researching how use of ICT tools can improve learning activities in mathematics by using inquiry as an approach in the development. Teachers and didacticians worked together to plan teaching using computer software to develop tasks for promoting inquiry into the mathematics and to develop pupils' understanding of mathematics and the ICT tools. The research was situated in a socio cultural framework, emphasising inquiry communities. I will describe a case with a teacher team, partly assisted by a didactician, working on task development, using the task with pupils, further reflecting and improving the task. I will illuminate characteristic features of the developmental process and how inquiry was used.

BACKGROUND AND RATIONALE FOR THE RESEARCH

There has been great effort to implement and utilise ICT tools to support learning in Norwegian schools. Much has been achieved in general, but little is implemented in specific subject areas, and in mathematics teaching in particular (Erstad, Kløvstad, Kristiansen, & Søybye, 2005). Hardly any use of ICT was reported in a recent evaluation of how teachers implemented the curriculum (Alseth, Breiteig, & Brekke, 2003) and the report also revealed weaknesses in the learning outcomes. Research in international comparison like TIMSS and PISA (Grønmo, Bergem, Kjærnsli, Lie, & Turmo, 2004; Kjærnsli, Lie, Olsen, & Turmo, 2004) report that Norwegian results are slightly below average internationally and below comparable countries. My own impression from contact with teachers discussing use of ICT is that many teachers lack knowledge of how to utilise ICT in mathematics and ask for good examples of use in order to understand how technology can support teaching and learning.

The ICTML project (ICT and mathematics Learning) [1] aims to meet this challenge and is designed to be a research and development project where teachers and didacticians [2] work together to inquire into how ICT can support and improve teaching and learning of mathematics by focusing on inquiry community. The project is implemented in close collaboration with the LCM project, Learning Communities in mathematics (Jaworski, 2005), with a common theoretical framework.

The research presented in this article is concerned with describing characteristic features of how didacticians and teachers work together, and if and how the teaching approach with ICT stimulates pupils' engagement in inquiry in the classroom.

THEORETICAL FRAMEWORK

The research is situated in a socio cultural framework, and with a learning community or rather an inquiry community as an overall theoretical perspective and

organising principle. The framework is inspired by Wengers' concept community of practice (Wenger, 1998), with the three modes of belonging, engagement, imagination and alignment as key concepts, and with alignment further developed into critical alignment (Jaworski, 2006b). The participants engage in the common activities, discussions and teaching, and develop ideas through imagination and align themselves with the community through critically testing ideas and developing understanding and new ideas. This has been conceptualised further as a community of inquiry (Wells, 2001; Jaworski, 2006a).

Inquiry means to ask questions, make investigations, acquire information or search for knowledge and 'dialogic inquiry' (Wells, 2001) furthermore willingness to wonder, seek to understand by collaborating with others. In the projects inquiry is a major focus in the work, with the aim to develop further into "inquiry as a way of being", characterised by willingness and engagement to inquire into all levels of the work in the learning community. In this context teaching is seen as a learning process; through inquiring into the various activities, mathematics and use of ICT, and as teaching is planned and carried through, this implies learning through the activities. The inquiry attitude also opens up for teachers and didacticians not knowing all the answers and engagement when new questions and problems arise.

For the ICTML project, inquiry encompasses the following areas in common with LCM: inquiry into mathematics, inquiry into mathematics teaching and inquiry into researching mathematics learning and teaching (Jaworski, 2006a). In addition for ICTML there is a strong focus on inquiry into computer software and how it can be utilised to stimulate and facilitate inquiry in mathematics teaching and learning.

THE ICTML PROJECT - RESEARCH AND DEVELOPMENT

The ICTML project has a key focus on inquiry into how ICT can support and enhance teaching and learning in mathematics. Although I see ICT as encompassing a broad range of software and hardware including calculators, our focus has been on computer software, and particularly software that is open and has potential for various ways of use. Such open software provides possibilities, or affordances, to ask questions, experiment, investigate – in short, use the tool for inquiry into mathematics and as part of the work we inquire into the software itself what it can afford for mathematics and how we can work around constraints. In the project we use mainly Excel and Cabri and more limited use of Grafbox, a graph plotting software.

The software itself does not create inquiry, the key is the way it is used, and how situations for learning are created. The design of tasks become important, and the development cycle a guideline for the work.

Didacticians and teachers work together and to some extent pupils work together and with teachers and didacticians to form learning communities in different settings within the project and engage in inquiry into mathematics and how mathematics can be represented in computer software.

Four schools participate in the project, with three of them also taking part in LCM. Activities in the project are workshops held at the University College, work in school teams in each school and in the team of didacticians at the University college in developing the framework for the project and responsible for the workshops.

At workshops in ICTML, two each term, we typically held the first part in the computer room, with some short introduction about features of the software, possible uses in mathematics or examples from teaching. This was followed by work on computers in small groups on suggested tasks or inventing new tasks and ways of using the software. After a break we had a summary discussion, looking at various examples from the work in the computer room and discussing further possible solutions and tasks. The teachers sometimes presented some of their approaches and innovative work from their teaching.

The teachers in the project form school teams for their own school and the aim is that they meet regularly to discuss relevant issues, and work together on their teaching: plan lessons, carry through, observe each others lessons and reflect on experiences from the classes and give feedback. This cycle of work in the schools fits well with the cycle of design in design research (Kelly, 2003; Jaworski, 2004) which has been further adapted to our developmental research. The teachers have the full responsibility for what is implemented in the classroom, in order for them to have the ownership and build on their experiences. But the intention is also that didacticians engage in the inquiry community with teachers.

METHODOLOGY

The research methodology is inspired by the design research cycle as described for the developmental work in the project. This research methodology has roots in various recent research methodologies, like action research, design research, lesson study and learning study (Jaworski, 2004). An important feature in the way the research is performed is to include teachers as partners in the research, taking part in discussions and it is hoped observe each other in the classroom, reflecting on experiences and noticing issues that arise during their work. In this way both didacticians and teachers in school take part in the research and development, and research is carried through in close cooperation. Furthermore, research is carried out on all levels in the project; including the didacticians' own work, the workshops and the work in schools.

The concepts 'affordances' and 'constraints' can be used to evaluate the implementation of ICT in teaching according to Kenewell (2001). Affordance is a concept introduced by Gibson, and further developed in psychology, to characterise features of the setting, objects or environment which provide potential for actions (Greeno, 1994). Constraints are conditions and relationships that provide structure and guidance. Constraints are not only negative; they are rather complementary to affordances and equally necessary for the activity (Kennewell, 2001). I see these

concepts helpful in describing characteristics of the development and experiences from the research.

In researching all levels of the project, it is customary always to video or audio tape our meeting, observations in class and workshops, and to write field notes.

A CASE OF TEACHERS' TASK DESIGN AND IMPLEMENTATION

I will give a narrative account of a case where a teacher develops some software applications for teaching about fractions in a class of grade eight pupils. The story is an example of how the development progressed and was tested and revised over two weeks in April 2005 cooperating with the school team and myself as a didactician.

I know the school from a previous project, but two teachers, Richard and Victor, were new to me in this project. They had some experience of computers before, they did not characterise themselves as experts but enthusiastic to learn more. The third teacher in their school team, Otto, also took part in the meetings, mainly as an adviser giving input and suggestions for further work. Otto was also appointed in the project as advisor for schools and to contribute in the workshops. He has a extensive knowledge and experience of using ICT and had worked with me in a previous project. I had met the three teachers both in the workshops at the university college and in previous meetings in the school team the autumn before this design case. I had visited the class so the pupils had seen me before.

The three teachers had set themselves the goal for their work in the project to develop applications in Excel to support various topics in the curriculum like fractions, area and volume, percentages and so on. They aimed at developing a library for their school with tasks for using an inquiry approach, experiments and investigations for the pupils in order to develop understanding of the specific topic.

In late March it was some time since I had been with them in their school and approached them with a suggestion for some dates I could come. The reply revealed they were quite busy and only one of the days I suggested would work for them. But Richard told they really wanted me to come. With some rearrangement from both sides we managed to adjust the time schedule to make it possible for me to attend a series of lessons, three days in April and some meetings to discuss the work. In the following I describe events from work in one class and the school meetings.

Richards' class was just about to start on a new topic, fractions, decimal numbers and percentages and he decided to make some tasks for that using Excel. The first lesson in the series started in the regular classroom. The teacher gave an introduction, how to find the computer file with the task and how they could proceed through the different parts by moving on to the next sheet in the Excel document. Richard had prepared the tasks in Excel late the evening before. In his work on Excel he used features of the software that he learned about in a previous meeting in the school team some months before.

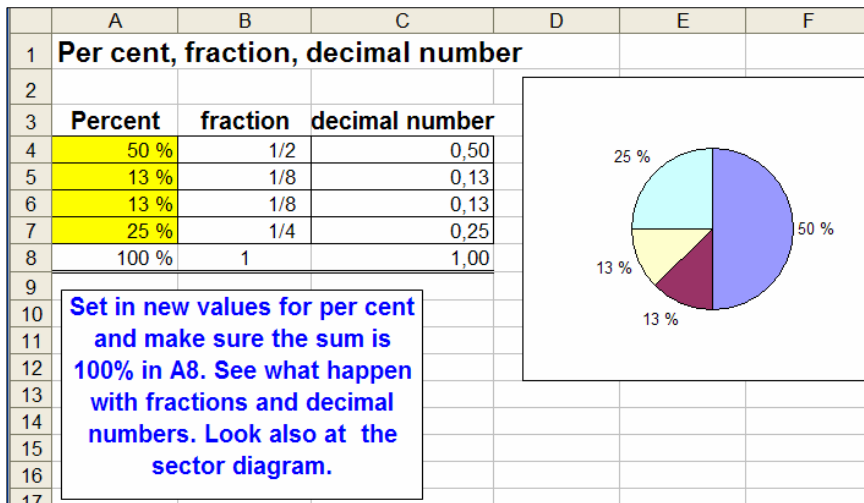


Figure 1 Task on per cent, fractions and decimals

After this short introduction in the classroom, the pupils moved over to the computer room next door and started working. The first task is shown in Figure 1. The idea is to insert numbers in column A for percentage, to make the sum 100% and observe the corresponding numbers in columns B and C and the diagram. The task was made by utilising the features of the spreadsheet to hide and lock the formulae and protect the sheet so only the cells A4- A7 are open for the pupils' input. In addition a diagram to show the parts and some formatting was used for good layout. The task tells the pupils to put in new values for per cent that add up to 100% and observe what happened in the next columns containing fractions and decimal numbers.

The next sheets in the first Excel document were of a similar kind, comparing just two items at a time. The pupils were then asked to investigate to see connections and write about what they found in a textbox provided next to the task on the sheet, and possibly find some kind of rule. The pupils struggled to write down their answers. I observed the lesson; audio recorded some pupils working and collected copies of their Excel-files. Often when Richard discussed and listened to their explanation he encouraged them to write it down – “Write it, yes write what you just said”. The notes in their documents were later used in discussion in the class. During the work in the computer room, the two other mathematics teachers engaged in the work, came in to see how it all went on in the class, and assisted as helping teachers.

After the lesson finished we had a meeting in the school team, the three teachers and me. Otto gave his first reaction – “it is far too much, too many items to compare” and he would like some more questions – for example what percentage leads to 4 parts, 5 parts, 8 parts and so on. The following sheets were less complicated. Experiences from the class were discussed with reference to what had been observed, alongside with details of how to make the questions accessible for the pupils. Ideas for how not to make too complicated comparisons and further improvements of the spreadsheet tasks were thoroughly discussed. Some constraints in the way Excel makes the graphs was discussed, and possible ways to get around them to facilitate

just the kind of illustration the teachers wanted, with a square divided into 100 and parts of it in different colours.

From the discussion I noticed we inquired into various areas of the work, the pupils' work, how they managed to find out and write their explanation, the task how it worked for the pupils and possible improvements to make the questions more accessible, and the constraints and affordances of the spreadsheet for the particular tasks involved. It seemed this made the teachers aware of the pupils' understanding.

Excitement over the experiences and our discussion was clearly expressed. Richard said: "This is great fun. I will enjoy working more on it. I will make something; it is always too difficult in the start". Although it was not planned in our e-mail correspondence, we agreed that I should come back two days later and the teachers found ways to rearrange bookings to get the computer room available. Richard wanted to follow up with more tasks using the ideas we discussed in the meeting.

Two days later Richard had made a new Excel-file, with tasks to compare per cent, fractions and decimals. Other tasks were introduced to make fractions giving numerator and denominator, with constraints on denominators, more questions and space for the pupils to write about their findings.

Figure 2 shows the first task in this new file. In this task the pupils are asked to write down all percentages they find that give as denominator the numbers 2, 4, 5, 10 and 100 and to see if they can find a connection.

The task is similar to one in the previous lesson, but the questions more constrained, taking into account the discussion from the school meeting two days before. I noticed some confusion occurred because the spreadsheet immediately simplified for example $2/4$ into $1/2$, and that made one of the tasks difficult to use in the current mode. Then the question came up if it were possible to turn this feature off. Richard commented he did not test all this before the lesson, and he hardly had time since this was prepared in the evenings after our school team meeting. But this became a challenge for some pupils who found out about formatting of fractions to some fixed denominators and then managed also to utilise this in their solution. A girl had written a long list of fractions that gave 100ths and 10ths, and described the numbers as odd-tens and pair-tens, which is a correct description of the solutions, but not common way to express this. There were some concerns also to the wording "equal fractions" meaning "equivalent fractions" or "equal valued fractions" as expressed in Norwegian.

	A	B	C	D	E	F	G
1	We make fractions						
2							
3							
4	per cent	fraction	decimal number				
5	55 %	11/20	0,55				
6							
7							
8							
9							
10							
11							
12							
13							
14							
15							
16							
17							
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19							
20							
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22							
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25							
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27							
28							
29							
30							

Figure 2 Next version of task

From the video it seems most pupils were deeply engaged and discussed their solutions. Some were challenged because of limitations in the spreadsheets' mode of working. During the work in the computer room Richard commented: "I think about Vygotsky all the time – I give them small hints and then they can make it – but they would not have made it without." He referred to the zone of proximal development and how important the small hints were, but not just to give them the solution (Vygotsky, 1978).

A week later, the class had another session in the computer room. This time Richard had developed further an idea that we had discussed briefly in the meeting after previous lesson, inspired by the problem with fractions that were simplified automatically in Excel. The idea was to get many equal valued fractions visible at the same time, but without having to format to a certain denominator. He got help from a colleague with more experience of spreadsheets to make some if-then tests to make it work. Again his mathematics colleagues attended parts of the lesson to observe the pupils' work and assist him.

Figure 3 shows the layout of the task. The idea is for the user to write a fraction in the middle (yellow coloured cells), numerator and denominator in different cells, and see what fractions appears in the other marked cells and write about the findings and possible explanation.

	A	B	C	D	E	F	G	H	I	J	K	L
1	Equal fractions					Set in numerator and denominator in the yellow cells Look at the fractions that appear in the green cells. Can you see a connection?	I see this connection: The fraction in the yellow window is equal to the others but it has larger numbers, the upper fractions is equals all the others and it is simplified so that the numbers are divided by two and so it is further down. If we start on the right hand side bottom and goes zigzag up then it is the numbers in the fraction is divided by 10, 9, 8, 7, 6, 5, 4, 3, 2, and it stops by 1 in the middle.					
2												
3												
4				18								
5				231								
6												
7			12									
8			154									
9												
10				36								
11				462								
12												
13												
14												
15												
16												
17												

Figure 3 Equal fractions

Asle and Nina worked on this, tried various fractions and observed other numbers appeared, for example 100/200 - Asle commented: "This is half". I observed them, heard him and asked "What is half?" Asle "They are going to be equal, completely equal". They worked on it and tried more numbers in the cells, for example 36/462. Nina: "I do not understand anything." And Asle was quite determined about half of something. It turned out he compared just the separate integer numbers (in the cells), and did not see the figures as fractions displayed in the coloured areas. The communication between pupils were just pointing and saying "this is" and "that is", and not by using mathematical terminology with numerator and denominator in their

comments to what they could see. They called on Otto to come and help them. He posed some more questions: “What is the largest?” pointing at the fraction in the middle and the one on the top”. The girl answers, OK they have the same value. Otto: “...then write it down”. They worked on writing, the girl said “This is half and this is the third and to the right is the sixth of the number in the middle”. And commented further: “This is fun!”

Asle and Nina asked Richard to show him their written explanation. They discussed with him, still coming back to their way to describe the connections referring to two integers separately, not as a forming a fraction. However, after some more discussion they finally managed to write a correct description which is shown in the Figure 4.

The next sheet was the same as the first one, but with the challenge to find a fraction that would give fractions for all the other marked fields. In order to solve this they had to find more connections and use that information. From their solution file I found they had also solved this. There were more tasks on the subsequent sheets, dealing with prime numbers and factoring. There was also explanation of how to make the fraction sheet and a task to make their own set up challenging the pupils to explore if-then sentences.

I also briefly observed some other pupils and found they had fewer problems than observed with Asle and Nina. Several pupils gave a correct solution and some also found the solution to the next problem, to put in the fraction in the middle to get fractions in all the boxes made for it. More observations were reported from the other teachers in the school meeting just after the lesson and they confirmed my impression. Victor commented he helped a pair of pupils to focus on one new fraction at a time, saying:”Look, when can you see something coming up here?” In this way he stimulated a more systematic way of finding connections. We discussed both pupils’ solutions and the way Richard made the spreadsheet. After explaining what and how he found out how to make it, he said he discovered really it was quite easy and he felt he succeeded with this task. Later after the lesson Richard commented again:”This is inspiring, I would not have made this if you had not come today. This pushes me – but it is really so interesting. When can you come back? Just send me a message when you can.” We planned some further lessons with me participating in the class, observing and discussing with the school team.

RESULTS FROM OBSERVATIONS

The experiences and challenges from the work I observed and the discussions and challenges we had together also were very inspiring to me. I experienced being part of their developmental work and although the one teacher did the major part of the planning and development of the Excel files, we all contributed by discussing technical solutions and creating new ideas. In this way the school meetings, working together in the class and other contacts are firm expressions of a learning community. We did not just discuss what we already knew, we developed further knowledge together and all in the school team took part in the work in Richards’ class. Ideas did

grow as we worked together, finding new approaches for the presentation of tasks, for example limiting which numbers can be used as denominators in exploring fractions.

I can see in this work an attitude of inquiry, moving towards a characteristic inquiry “as a way of being”. Inquiry into how to present tasks for experiments and investigation for the pupils, and inquiry into how to utilise functionalities in Excel for achieving what was wanted in the tasks. Problems during the lessons became challenges for further inquiry into how to utilise affordances and to get around limitations in the software, or just utilise functionalities to constrain features in the tasks to make them more accessible and clearer for the pupils to work on.

Observations of pupils’ work with the tasks revealed weaknesses in the tasks, but also revealed where pupils’ thinking were not well developed concerning fractions, and percentages. To write about their observations and find rule appeared quite challenging for pupils. They seemed to find it easier to explain orally, but were then challenged by the teachers to write down what they just said. This input can be seen as an affordance from the teaching environment, provided by the design and teaching.

In the case I have described here I did not observe the summary class discussion, but from later observations in the same class I have seen that these written explanations form an important part of the development for the pupils and the class. The explanations were utilised in the class discussion, presented by the pupils themselves in the class, commented upon and in this way helped to summarise and consolidate the knowledge they developed. I see this as an affordance built into the task for supporting the pupils’ development.

CONCLUSION

Although it was not an explicitly expressed aim at the start of the work, this case had the major steps in the inquiry cycle built into it. There was some planning from the start: the choice of mathematical topic, fractions and use of computer software for introducing tasks with the purpose of pupils investigating and learning. This first major planning was done by the class teacher. The school team observed and later discussed experiences and gave feedback. Affordances and constraints in Excel were revealed and utilised in the design of tasks, elements from previous meeting taken up and further developed. The observations and discussions confirm the teachers learning and enjoyment, and the impression is encouraging when we observed the pupils’ work.

NOTES

1 The project is supported by The Research Council of Norway

2 Didacticians in this context refer to researchers and doctoral students at the university college

REFERENCES

Alseth, B., Breiteig, T., & Brekke, G.: 2003, *Endringer og utvikling ved L97 som bakgrunn for videre planlegging og justering - matematikkfaget som kasus*. Notodden, Norway: Telemarksforsking.

- Erstad, O., Kløvstad, V., Kristiansen, T., & Søybye, M.: 2005, *ITU Monitor 2005. På vei mot digital kompetanse i grunnopplæringen*. Oslo: Universitetsforlaget.
- Greeno, J. G.: (1994). Gibsons Affordances. *Psychological Review*, 101, 336-342
- Grønmo, L. S., Bergem, O. K., Kjærnsli, M., Lie, S., & Turmo, A. 2004, *Hva i all verden har skjedd i realfagene? Norske elevers prestasjoner i matematikk og naturfag i TIMSS 2003*. Oslo: Acta Didacta, Institutt for Lærerutdanning og skoleutvikling, Universitetet i Oslo.
- Jaworski, B.: 2004, Insiders and Outsiders in mathematics teaching development: the design and study of classroom activity. In *Research in Mathematics Education Papers of the British Society for Research into Learning Mathematics*, vol. 6,1-22 British Society for Research into Learning Mathematics.
- Jaworski, B.: 2005, Learning Communities in Mathematics: Creating an Inquiry community between teachers and didacticians. *Papers of the British Society for Research into Learning Mathematics, Research in Mathematics Education*, vol. 7, 101-120.
- Jaworski, B.: 2006a, Learning Communities in Mathematics: Research and Development in Mathematics Teaching and Learning. In C.Bergsten, B. Grevholm, H. S. Måsøval, & F. Rønning (Eds.), *Relating practice and research in mathematics education. Proceedings of Norma05, Fourth Nordic Conference on Mathematics Education*. Trondheim, Norway: Tapir Akademisk Forlag.
- Jaworski, B. 2006b, Theory and practice in mathematics teaching development: Critical Inquiry as a mode of learning in teaching. *Journal of Mathematics Teacher Education*, 9, 187-211.
- Kelly, A. E.: 2003, Research as design. *Educational Researcher*, 1, 3-4.
- Kennewell, S.: 2001, Using Affordances and Constraints to Evaluate the Use of Information and Communication Technology in Teaching and Learning. *Journal of Information Technology for Teacher Education*, 10, 101-116.
- Kjærnsli, M., Lie, S., Olsen, R. V., & Turmo, A.: 2004, *Rett spor eller ville veier? Norske elevers prestasjoner i matematikk, naturfag og lesing i PISA 2003*. Oslo: Universitetsforlaget.
- Vygotsky, L. S.: 1978, *Mind in society. The development of higher Psychological processes*. Cambridge, Mass.: Harvard University Press.
- Wells, G.: 2001, The development of a community of inquirers. In G.Wells (Ed.), *Action, talk, and text: Learning and teaching through inquiry*. New York: Teachers College Press.
- Wenger, E.: 1998, *Communities of practice. Learning, meaning and identity*. Cambridge: Cambridge University Press.

TECHNOLOGY THAT MEDIATES AND PARTICIPATES IN MATHEMATICAL COGNITION

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The presence and intelligent use of digital technologies in mathematics education has awakened the interest for understanding new ways of conceiving mathematics and mathematical cognition. Cognition, especially mathematical cognition, can be understood in terms of emergence of successive and evolving representational systems. As a consequence, the presence of digital technologies in education must face this fundamental fact, as curricular structures eventually will be inhabited by these technologies. However, we cannot forget that a school culture always leaves significant marks on students' and teachers' values. Artigue (2005, p. 246) states that "these [previous] values were established, through history, in environments poor in technology, and they have only slowly come to terms with the evolution of mathematical practice linked to technological evolution". Thus, the school culture requires the gradual re-orientation of its practices to gain access to new habits of mind and to the new environments resulting from a serious presence of digital technologies. Consequently, we will present a new perspective with examples from our research to approach the new *problematique* made tangible by digital technologies. We particularly focus on the role of "dynamic mathematics" as an umbrella description of certain technology (both software and integrated hardware) that opens up a new exploration space for learners. We present data and analyses of students working with dynamic geometry software (such as Cabri and Sketchpad), and software that links multiple representations of function in interactive ways across networks (e.g., SimCalc). We refer to these more globally as *Dynamic Technological Environments* or simply DTE(s). Within such environments, students are capable of exhibiting new forms of expressivity associated with their explorations and new forms of understanding based upon the capacity of the environment to react to the actions proposed by the students.

THE MAIN ATTRIBUTE OF A DYNAMIC TECHNOLOGICAL ENVIRONMENT

We present the idea of a DTE by creating various descriptive constructs that can help researchers identify what particular features distinguish the variety of educational technology available for mathematics education today. In describing these features, we aim to begin to create a categorization scheme to aid researchers to investigate the impact of classes of technology vs. one specific technology at a time. Such attributes can help build epistemological descriptions of how technology in education has evolved. In doing so, we might be able to assess the potential impact and effectiveness of certain attributes of technological environments on the teaching and learning mathematics in classrooms today.

Attribute 1: Recursive Exploration Space. We begin by proposing that a DTE can be conceived of and used as a *recursive exploration space*. The complete action–reaction loop exhibited by the co-action of the student and the digital environment, forces cognition to be distributed in the space defined by the agent (the student) and the environment. The emergent distributed *intelligence* is made tangible by a responsive environment (increasing in agency) and the digital tools in front of the agent-role of the student.

The co-action developed between the agent and the environment generates the exploration space that we call a DTE—in other words, something that can be understood as a space in which students appropriate ideas that emerges from the co-action and absorb a recursive level of formalization for their explorations (Moreno & Santos-Trigo, in press; Moreno & Sriraman, 2006). In a dynamic technological environment, the general (of the knowledge) is embodied in a particular vision (of the agent). Dynamical technological environments (DTE's) empower students to offer avenues to understand generality and systematization from concrete forms of embodiment. So through interaction and exploration of dynamic constructions, students can make sense of the mathematical attributes of the structure of a geometric shape, figure or theorem, or the underlying structure of a system or family of functions (Hegedus & Kaput, 2004). Exploration is a gradual process that can be propagated through visual tools, or feedback, central to the computational affordances of the DTE. For example, with three points selected in Geometer's Sketchpad, you can do one of several Euclidean operations (constructs rays, segments, lines, angle bisector, an arc through 3 points, or a triangle interior). As we build on that construction, the user can *direct* the DTE through interaction with the construct and dynamic tools, or *be led by* the environment in the form of tools that become available through dynamic menus or visual feedback through “draggable” features such as hotspots (see Hegedus & Kaput, 2004). As the procedure of dragging and reflection is repeated many times, a mathematically cognitive recursive procedure is created that is analogous to a recursive mathematical procedure in an analytic sense. Such reflection leads to more refined mouse movements and subsequent reflections on the dynamic diagram that is being examined. We use the Euclidean construction as a very simple, yet prototypical example of this iterative procedure, which creates the *recursive exploration space* for the purposes of refined and supportive mathematical discovery.

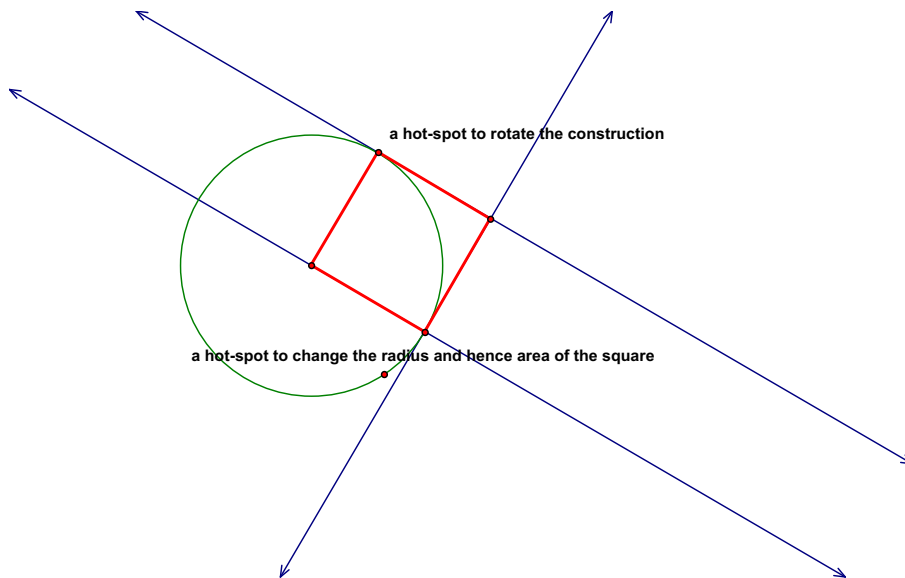


Figure 1. A Euclidean construction of a square.

The hotspot allows for a rich milieu or mathematical structure to be explored. For example, a Euclidean construction of a square could be built and then a central hotspot (i.e., one which is a parent to the whole construction) can be dragged to test the resulting construction (i.e., children of the parent inputs) to see if the square is preserved. In effect, the construction is invariant under dragging of the hotspot, i.e., the construction is a true Euclidean construction.

As we have already said, a DTE can be a space not only to explore inductively (which is an important path to discovery) but more importantly from our framework, a space co-extensive with possibly new ways of formalizations. For instance, let us describe briefly an example taken from exploration-work with student teachers.

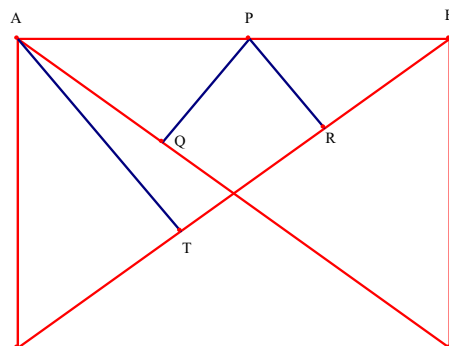


Figure 2

The problem they were exploring is the following: the sum of the lengths of segments PR and PQ (perpendicular to the diagonals of the rectangle) remains constant when P moves freely along the side AB (see figure 2). A DTE provides cognitive-exploring resources that are absent from a static environment. For instance, students were able to displace point P to the left until PR and PQ became PT, and this is a fixed segment with a fixed length. From here on, the exploration had a more precise goal as it was clear that if the sum of the lengths was going to be a constant, that constant was the length PT (see figure 3 below). Thus, the problem has lost some of its cognitive

demand (we might say that the *cognitive distance* between the student and the problem has diminished). The expressive capacity is enhanced both from the side of the student and from the side of the environment as well (made explicit as its responsiveness during the explorations).

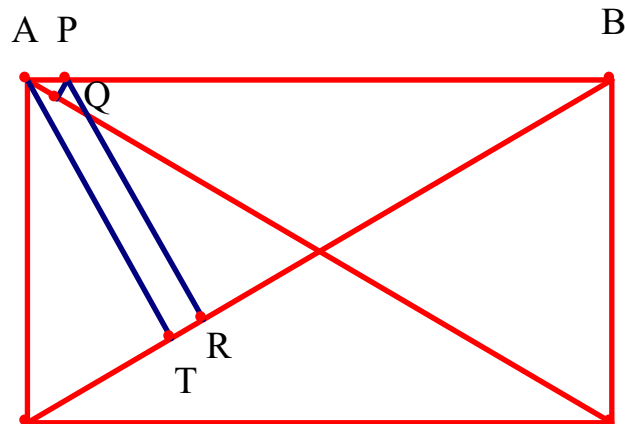


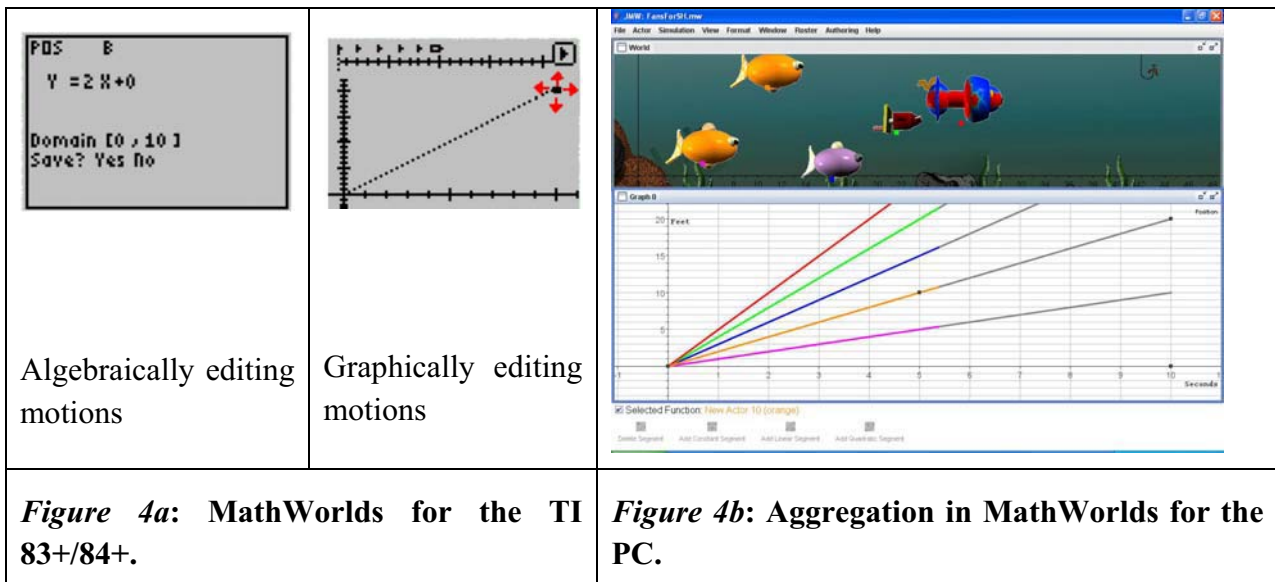
Figure 3

INTEGRATING MULTIPLE REPRESENTATIONS WITH NETWORKS

Attribute 2: Students can discuss general properties supported by the dynamical and executable semiotic representations embodied in the environment.

Students can create mathematical objects on hand-held devices (such as TI-Graphing calculators) and send their work to a teacher computer, which is projected on her whiteboard. Due to advances in wireless communication and interactivity between desktop PCs and hand-held devices, the flow of data around a classroom can be very fast allowing large iterations of activities to be executed during one class. But it is not just an advance in connectivity but in the development and application of curriculum that maximizes such an innovation. Our prior work created activities that allowed students to make functions in SimCalc MathWorlds on the TI-83/84+ graphing calculator that could then be aggregated by a teacher into SimCalc MathWorlds running in parallel on a PC, using TI's Navigator wireless network (see www.simcalc.umassd.edu for downloads of the software). Alternatively the teacher can shift the agency of the classwork to the students by allowing students to send their contributions at will. Such an action by the teacher, though, was not done in an arbitrary fashion (i.e., collect all work) but in a mathematically meaningful way. For example, each student is in a numbered group (say 1 through 5), and has to create a position function that can animate an actor (in a simulation) at a constant speed equal to their group number for 10 seconds. So group 2 can use SimCalc MathWorlds to create a function algebraically (see Figure 4), i.e., $y=2x$ on the domain $[0,10]$, or graphically by building a linear function, and dragging a hotspot attached to a line segment with left endpoint at the origin out to $(10, 20)$. When all functions are submitted by the students, or aggregated by the teacher, then a family of functions, $y=mx$ ($m=1$ to 5) will be displayed and publicly analyzed. This simple activity can be extended into an activity structure that uses group count-off indices in general to distribute mathematical variations across a whole class of students. This latest innovation expects more active participation by students since every student is required to contribute, but not only that, engagement is potentially affected even if a student does not contribute; the aggregate (displayed by the teacher


computer) exposes this and, potentially, errors that some students might have made upon analysis of the collection. This is a main working hypothesis and our study has investigated the reality of such a claim. This transfer from private or local work that students have done individually (or in a small group) to a whole class public workspace allows different forms of participation and analysis since students can contribute mathematical objects in non-verbal forms and make sense of their own work in a social context as well as generalizing about their contribution in relation to their peers' work. This flow of communication has restructured the social space of the classroom. Each student's individual contribution is a piece of the whole. And collectively the contributions expose the mathematical variation that exists. Students are co-acting with the software environment but also with each other when the collected work from the entire class is gradually displayed conjointly with certain pedagogical intentions in mind and analyzed mathematically. The technology becomes a partner with the teacher at a public level to support the emergence of ideas, support or refute conjectures made by students and guides, as well as be guided by, the software at a local, personal level, as students interact and explore dynamic links between graphs, simulations, and each other's thoughts. This has led to student participation in a variety of physical and oral forms (Hegedus, 2006).




In the following analysis of a classroom episode of conducting such an activity as described above, we note the physical expressivity as well as the social interaction that occurs to make sense of the variation within the mathematical structure. Again the mathematical structure is understanding the parameter m , in $y=mx$, potentially a simple variation in our mathematical example, but a prototypical example in our example set. Here we vary across function, to highlight publicly the *stretch of variation* across a family of functions through 1) aggregation computationally, 2) aggregation cognitively in the sense that a student has to expect what she sends will meaningfully show up i.e., what she will see is an aggregation of her work with her classmates work in a parallel version of our PC software via TI-Navigator Learning system. Variation across a family of functions is distributed via several personal, and identifiable contributions from each student, or group of students. In this example, group 1 will contribute $y=x$, group 2 will contribute $y=2x$, etc. Each contribution will show up publicly, i.e., can be displayed and examined at a public level. Now,

when one function is “incorrect” or debated in some way, a teacher can correct it based upon some public analysis or a student can edit their own function from their own personal location, i.e., when they edit their function on their TI-83Plus/84Plus, it will “instantaneously” show up in the networked computer version of SimCalc MathWorlds, displayed publicly to the class via a projector.

In this example, the activity is structured such that students are constructing motions within the software which are driven by linear functions. They have created a motion, by editing a linear function either graphically or algebraically, where the velocity is equal to their group number on a set domain, $[0,10]$. This is important in the structure of the activity because simulations are at the heart of the software. These graphs or motions are created in SimCalc software on the TI-calculator and aggregated via TI-Navigator Learning System into SimCalc parallel software on the PC. Work has been collected but not displayed yet; the teacher is asking what the students expect to see in the set of graphs for the entire class. The software does not show all collected functions at once. We expect the teacher to make a pedagogically adept decision (following research-informed curricula materials); such decisions are strategically posited to seek a constructive learning trajectory in classrooms where students progressively see their work compared to their classmates’ work. The comparison of their work to their classmates’ contributions is a mathematically meaningful act. The following transcript exemplifies how non-verbal communicative acts are used for students to communally “see” and listen to how structure can be expressed, conceptualized and discussed at a social level (Hegedus *et al*, 2006). The aggregation of functions is done as an external artefact for discussion. The display of $y=mx$ with m varying as a slope of a family of linear functions has been described as a collection of motions spreading outwards in SimCalc MathWorlds software, as well as a “fan” of functions describing the graphical representation via a gestural metaphor - a physical splayed-out finger representation - to an external construction to “physically” represent a family of linear functions, i.e., a collection of pens. Below you will see how some students realize and demonstrate how they want to describe how their understanding of such a family of functions through physical, metaphorical and discursive methods of expression.

<p>Teacher: Alright, what would it look like? What the graph of all five, all five lines look like? What should it look like?</p> <p>S: a spine</p> <p>B: like my description</p> <p>Teacher: what was that?</p> <p>B: it goes like this.</p> <p>K: like this. {This time K picks up pencils and begins to re-create the graphs.}</p>	 <p>{K starts making a gesture with her whole arm and B, who is seating diagonally and in front of K does not see K, but begins to make a similar gesture.}</p>
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<p>B: I don't know. {<i>B turns around and sees what K is doing.</i>} Yeah like that.</p> <p>Teacher: Yeah like that.</p> <p>K: I need more pens. {<i>K is trying to count out the correct number of pens. She needs five to be accurate.</i>}</p> <p>Teacher: But with five of them K?</p> <p>B: It'd be like a fan.</p> <p>K: Yeah. Yup. {<i>K is still collecting pens and pencils from her friends.</i>}</p> <p>Teacher: What's that?</p> <p>K: like a fan</p> <p>Teacher: like a fan, alright.</p> <p>K: like this. {<i>K now has five pens or pencils and is displaying them for the class to see.</i>}</p>	 <p>{K now has five pens or pencils and is displaying them for the class to see.}</p>
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ANALYSIS

Attribute 3: Executability. This is part, as well, of the process of *socialization* of the tool, needed for being embodied into a DTE. From “being guided” by the tools, obeying almost blindly the syntax of the software, the students recursively access a level wherein they guide the tool leading eventually to a process of *co-action* developed through their activity (private, personal, and/or public). *Co-acting with the environment* leads to a re-definition of the tool itself (Hegedus, 2005). Needless to say, this is crucial in the classroom and an educational goal second to no other one. From being a cognitive prosthetic device, the digital devices become a substantial part of the *intelligence* of the students. It is the executability of the semiotic representations that acts as the substrate to this very complex and long process. Let us go a bit deeper into the nature of the executability of the semiotic representation. As Duval (2006) has very aptly written with respect to mathematical learning:

...there is a basic difference between mathematics and other domains of scientific knowledge. Mathematical objects, in contrast to phenomena of astronomy, physics...etc., are never

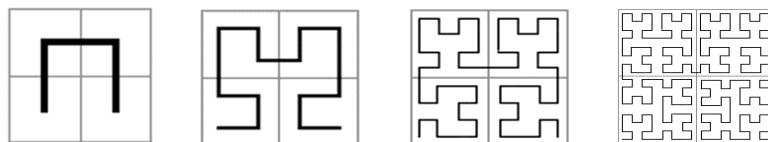
accessible by perception or by instruments (microscopes, telescopes, measurements apparatus). The only way to have access to them and deal with them is using signs and semiotic representations...[for] how can [the learners] distinguish the represented object from the semiotic representation used if they cannot get access to the mathematical object apart from the semiotic representation? (p. 107)

This is a deep and very important assertion. The feeling of not knowing what mathematics refers to—which so many students experience—is due in great measure to be placed in that position: a discourse without subject matter.

We hasten to say we are not adopting a Platonic posture, reducing the symbolic activity so essential to mathematics to a *naming role*. Far from it. But how then can we answer Duval's deep question? As we have shown with our examples, the executability of the semiotic representations provided by the medium generates a dynamic experience that goes through a labeling-revealing intentionality-generation of field reference process. This is almost inaccessible for most students facing the static traditional page where instead of appreciating the crystallized symbols (where action is living and encoded) they see only frozen inscriptions. Didactics, today, must work at the level of the intentional field that goes from the paper, as a frozen environment, to the screen that constitutes the space of digital representations where the sign acquire its digital nature. This redefines Duval's problem. Again, we are not saying that writing, as we have known it for centuries—as an instrumental force to create modern societies and new ways of thinking—is insignificant. What we are suggesting is the adoption and adaptation of a cognitive strategy based on the transforming power of the executability of semiotic representations as they live in the domains of abstraction provided by dynamic environments. The epistemological nature of executability belongs to another realm we are not entering today.

However it is pertinent to mention that mathematics education as a field of scientific inquiry can be assimilated to a certain form of applied epistemology. The nature of cognition and the nature of knowledge cannot ignore each other.

Let us now introduce another example to indicate the cognitive force of executability to gain access to mathematical knowledge. As part of the scaffolding process to conceive of a technology oriented-curriculum, a new way to conceptualize mathematical objects and their study is needed. Formalization is relative to the medium in which it takes place, thus there is a need to reflect on the new ways students may have to *prove* mathematical assertions in the classroom. Take for instance, the Hilbert space-filling curve:



This result brought with it fresh air. In a social and cultural medium where mathematics was trying to be *reduced* to the rules of number (the reader should be made aware of the so-called Arithmetization Program developed during the second part of 19th century that expelled movement from mathematics) a proof that made explicit a visualization process was more than welcome. However the static medium at disposal forced an analytical line of reasoning. To prove it along

classical lines is an intricate task. But what happens when one turns the result into an executable one (Santos-Trigo & Moreno, 2006)? To argue in favor of the validity of the theorem, we translate it: given a (screen) resolution, there is a step in the recursive process that generates the curve that fills that screen. This is a way to empower students to have access to deep, properly translated results. And most significantly, this empowerment is feasible due to the existence of executable semiotic representations of the Hilbert result. Conceptual transitions are always difficult as people become more sensitive to the gains but perhaps more to the losses. The new is a promise, but the present has been conquered at the expenses of deep efforts. is an active value. Let us illustrate this situation with the viewpoint of von Koch facing the arrival of the analytic approach in the hands of Weierstrass:

Until Weierstrass constructed a continuous function not differentiable at any value of its argument it was widely believed in the scientific community that every continuous curve had a well determined tangent...Even though the example of Weierstrass has corrected this misconception once and for all, it seems to me that his example is not satisfactory from the geometrical point of view since the function is defined by an analytic expression that hides the geometrical nature of the corresponding curve...This is why we asked ourselves—and we believe that this question is of importance also as a didactic point in analysis and geometry—whether one could find a curve without tangents for which the geometrical aspect is in agreement with the facts.

CONCLUSION

In summary, we have offered a few prototypical examples from Dynamic Technological Environments to support our argument that the structure of mathematical ideas can be re-conceptualized by teachers when using technological tools in a co-active way. In so doing, they offer a new environment for students to explore and discover mathematical reasoning. Ruler and compass provided a mathematical technology that found its epistemological limits in the three classical Greek problems (trisection of an angle, duplication of the cube, and the nature of π). Ruler and compass and in general, the mathematics developed within the space of the paper, embody certain normative criteria for validating mathematical knowledge. And more general, they are an example of how an expressive medium determines the ways to validate the propositions that can be stated there.

In order to further develop the scheme of attributes of DTEs we ask: What kind of propositions and objects are embodied within dynamical mathematical environments? The way of looking at the problem of formal reasoning within a dynamical environment is of instrumental importance. The DTE can lead the students in their inquiry, as well as be led by inquiring students as they accommodate the experience the environment offers. Finally, software such as dynamic geometry or SimCalc MathWorlds not only mediates classroom discussion in rich ways through the expanded role of gesture and metaphor, but also takes part in the learning process by being a non-verbal communicative “voice box;” offering feedback to assist the teacher in what to focus attention on and hence support classroom discussion and learning.

Once we ascribe agency to tools and technologies, we have an opportunity to rethink our relation to technology in terms of *partnership*. While the notion of intelligence distributed across tool and user

is far from new, going back at least to Dewey, by adding the notion of tool-agency, we may be able to examine our relation with technology in a deeper way that helps us understand the impact of technology in mathematics education in more systematic ways, and its role in the development of mathematical cognition.

REFERENCES

- Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61(1-2), 103-131.
- Hegedus, S. (2005, September). Dynamic representations: A new perspective on instrumental genesis. In M. Bosch (Ed.), *Proceedings of CERME4, The Fourth Congress of the European Society for Research in Mathematics Education*, Barcelona, Spain: Ramon Llull University.
- Hegedus, S., Dalton, S., Cambridge, L., & Davis, G. (2006). Patterns of participation in networked classrooms. J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds.), *Proceedings of the 30th International Conference for the Psychology of Mathematics Education* (Vol. 3, pp. 257–264). Prague: Program Committee.
- Hegedus, S., & Kaput, J. (2004). An introduction to the profound potential of connected algebra activities: Issues of representation, engagement and pedagogy. In *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 73–80). Bergen, Norway: Conference Committee.
- Moreno, L. & Santos-Trigo, M. (in press). Democratic access to powerful mathematics in a developing country. To appear in L. D. English (Ed.), *Handbook of International Research in Mathematics Education*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Moreno, L., Sriraman, B. (2006). The articulation of symbol and mediation in mathematics education. *ZDM*, 37(6), 476–486.
- Santos-Trigo, M., & Moreno, L. (2006). Students' development of mathematical practices based on the use of computational technologies. Paper to be presented at the 17th ICMMI study "Digital Technologies and Mathematics Teaching and Learning." Hanoi University of Technology, Vietnam, December 3-8, 2006.

THREE DIMENSIONAL CONSTRUCTIONS USING AN ABSOLUTE FRAME OF REFERENCE IN A COMPUTER SIMULATED 3D SPACE

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The 'building blocks' computational environment combines multiple representational registers in such a way that graphical relations are tightly linked to theoretical geometrical relations. The meanings constructed by primary school students in relation to the 3d coordinate frame of reference as they were manipulating 3d cubes through everyday-language commands are reported and discussed here. It could be said that the 3d cubes functioned as signs both at an iconic and an indexical level, giving way to conventional symbolic signs, connecting in a meaningful way visual spatial positions to an absolute frame of reference.

THEORETICAL FRAMEWORK

In this paper we report research aiming to explore the mathematical meanings constructed collaboratively by 11 year-old students concerning the mathematical notion of absolute coordinate frame of reference in a computer simulated 3d space. The computational environment used is an applet developed by the Freudenthal Institute (Jonker & vanGalen, 2004) that builds upon students' everyday activities and experiences with manipulables in order to support mathematical activity in a game-like 3d environment. Students use virtual cubic 'building blocks' so as to build 3d constructions which are equivalent to a plotting of points (the blocks) expressed in the form of 3d coordinates.

Coordinate systems are not a subject taught in lower levels of compulsory education in Greece at least, either in two or three dimensional planes. Coordinate geometry is an abstract way of positioning and orientating in the plane or in space, since it refers to a mathematically organised and absolute frame of reference. This way of referring to space requires the ability to disassociate from an egocentric experiential viewpoint, e.g. positioning an object right or left in relation to a person's position. This is rather difficult for students up to the end of primary education at least (Piaget & Inhelder, 1957, Lawler, 1985). To make things worse, the pre-technological means of representing and using coordinate systems at this level were mainly graphical since analytic geometry is taught at higher levels anyway. It is therefore not surprising that although the mathematical formalism of 3d coordinate systems is a powerful representational register for mathematics, it is often used in activities that seem meaningless to students. Digital technologies could provide us with means to reconsider the ways in which the notion of coordinates could be much better exploited for mathematical meaning making (Kynigos, 2002). However, relatively little learning research has been carried out pointing to how interactivity, dynamic

manipulation and experimentation in 3d game-like environments can be a versatile vehicle for enhancing the conceptualisation of 3d space in a mathematically meaningful way.

In the ‘building block’ computational environment mathematical knowledge is intrinsically linked to a construction process as students may position 3d cubes only by defining their 3d coordinates either by numbers or by using variables. Bypassing in a way the absolute formal way of representing 3d coordinates, cube is conceived more as a mathematical entity and a transitional object that represents a point in the 3d coordinate system. Thus it could be said that the focus is not on the concept of coordinate in its ‘pure’ abstract mathematical form but in the process of defining a place in 3d space using an absolute coordinate frame of reference. Construction with ‘building blocks’ goes beyond the visual recognition of spatial relations as the role of 3d representations is not just implicit and supporting to construction of the intended mathematical concepts but interwoven with them. The users of the ‘building blocks’ environment come in contact with three representational registers: 1) a discreet number of everyday language commands, 2) mathematical notation (e.g. whole numbers) and formalism (variables, 3d coordinates), 3) a 3d representation of manipulatives.

Our approach to studying students’ learning integrates constructionist (Harel & Papert, 1991), and socio-cultural approaches (Crook, 1994). Students construct their knowledge and understanding of the world not just through direct personal experience and discovery, but also through the intellectual sharing, challenge and support of those around them. Simultaneously students gradually form their own mental constructions of the essence and the functionalities of the tool and develop schemes of use which are often quite different to those intended by the designer of the tool or the learning environment. The construction of mathematical meanings as a result of students’ interaction with a specific learning environment raises (among the others) semiotic issues related to screen representations (Mariotti, 2001). In our study, we analysed students’ dialogues and actions, focusing on the interplay between the notion of absolute 3d coordinate systems, its spatial representation through cubes in a 3d computer environment and the utilisation of the software construction processes.

RESEARCH SETTING AND TASKS

The research reported herein took place in the computer lab of a public primary school. Four 6th grade students used the ‘building blocks’ computer environment and worked collaboratively in a set of activities which lasted 6 hours in total over 3 days. The first task which was considered introductory comprised the construction of a predefined ‘building’. The second task comprised the construction of all possible ‘buildings’ if only the two facets of a ‘building’ are known and given that each two opposite facets are the same. The third task included a half – made building that students were asked to change so as to create something of their own choice. Before introducing students to tasks, there were some preparatory examples provided in

order to introduce students to the game and to the way they could manipulate the 3d cubes.

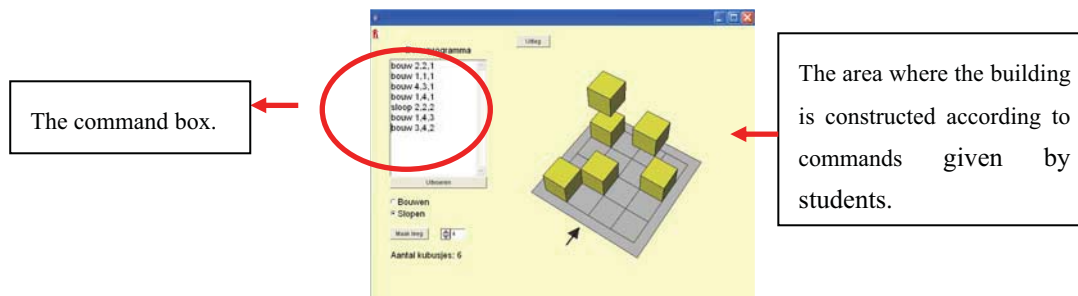


Figure 1. The interface of “Programming Blocks” game

METHOD

In our research we used a design-based research method (Cobb & all, 2003) which entailed the ‘engineering’ of tools and task, as well as the systematic study of the forms of learning that took place within the specific context as defined by the means of supporting it. Initially the aim was to develop the software and the task in order to improve the educational process and to bring about new forms of learning, based on prior research and our theoretical framework. In retrospect, the research method that we used aims both at improving the initial design and at resulting in a situated understanding of the relationship among theory, designed artifacts and practice.

Espousing the role of naïve and participant observer the researchers took a generative stance allowing for the data to structure the results and to pilot their analysis. It should be stressed that we are not aiming at final and self – evident answers but at focusing better and at reformulating our initial questions. A team of two researchers participated in each data collection session using one camera and two tape-recorders. One researcher was occasionally moving the camera to all groups to capture the overall activity and other significant details as they occurred. Background data were collected (e. g. students worksheets, observational notes) and all audio-recordings were analyzed verbatim. In analyzing the data we first looked for instances where meanings related to 3d coordinates were expressed by the students. However word episodes were meaningless if they were not related to the sequences of actions that students carried out while constructing their ‘buildings. Thus in the analysis that follows we minimized the use of word episodes as it seemed to us that carefully selected printscreens could be more illuminating of students’ construction strategies.

THREE-DIMENSIONAL CO-ORDINATES

The students that participated in our microexperiment were not acquainted formally to the notion of co-ordinates in the framework of their official curriculum neither in 2D nor in 3D plane. Initially we observed that students had difficulty in systematizing and mapping the 3 co-ordinates that defined the place of each building block with the

predetermined way of ordering each coordinate in their commands. The following word episode is indicative of their uncertainty as far as the order of co-ordinates is concerned (lines 41, 42). Students are puzzled if the sequence of numbers should be 3 (x coordinate), 2 (y coordinate), 3 (z coordinate) or vice versa. In the end they decide to input 3 for the x coordinate, 3 for the y and 2 for the z coordinate.

- 40 S4: ,3 ... 2,3...2 isn't it?
 41 S3: 2,3 or 3,2?
 42 S4: 3,2 isn't it?
 43 S3: What about this 3?
 44 S4: You mean 3,3.
 45 S3: 3...3,3...
 46 S4: ,4 3,3 ,2
 47 S3: 3,3 ,2
 48 S4: Hey, this one doesn't fit there.

The interesting point here is the way intuitive descriptions (e.g. the order of numbers that corresponds to the three coordinates) and courses of action are gradually formalized to what is 'mathematically acceptable' in a meaningful way. Constructions with building blocks go beyond a simple visual recognition of spatial relations. They require a systematicity that draws upon mathematical conventions. In other words mathematical conventions are not abstract theoretical constructs but acquire meaning through use of visually 'concrete' computational tools.

The essence of any reference system is the existence of a stationary point and of a sense of direction. In 'building blocks' environment each cube's position should be conceptualised as composed of 3 consecutive numbers measured from a fixed point of reference with a certain direction and with a certain order referring to each dimension. In our research students used a reference point as a means of locating the position of building blocks and a stable direction (left-right, bottom-up, front-behind). The fixed direction of construction in the 'Building Blocks' environment didn't seem to cause problems to students, probably because there were accustomed to have the left hand side area as a place of origin from their reading and writing activities. However the stationary point -the left hand front corner of the squared pat- had occasionally changed according to their view. For instance when they have tried to define a series of blocks when they were seeing their construction from the right view (figure 1), students have confused the length (x co-ordinate) with the width (y co-ordinate), creating a column in the x axis while it should be constructed in the y axis. Nevertheless the cues provided by the software (a black arrow indicating the front side of the construction) helped children realize the reason for the mis-correspondence between their expectations and what was actually constructed by the software.

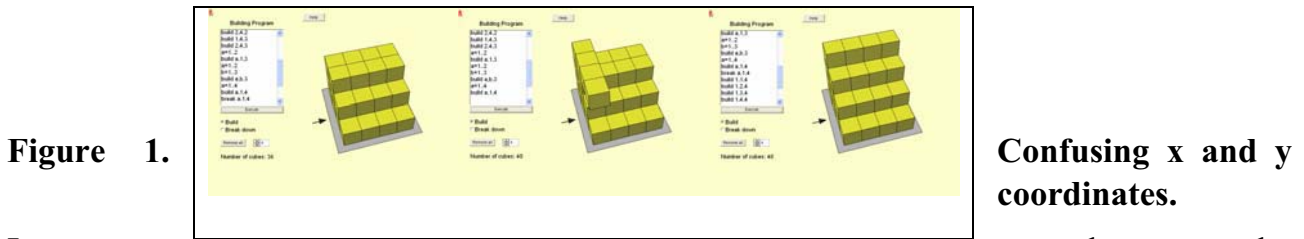


Figure 1. In accordance to the above finding is the displacement of the place of origin of the co-ordinate system. In certain activities, especially in cases where the base pad was more dense (divided in many cubes) students had calculated the position of the cubes they wanted to build having as a start the left hand front bottom corner of the building they have created and not of the square pad. For instance, in figure 2 students wanted to build a second level in the central building of the pad, but as a result of an arbitrary point of origin for the estimation of the cubes coordinates, the row they have created is placed in an unpredictable position for the students. It seems that children were swinging between the arbitrary and the fixed co-ordinate frame of reference. Once again the constraints as well as the new possibilities of the building blocks environment gradually shaped students construction strategies towards more formal ones that were not restricted to specific views or to other egocentric ways of conceiving space.

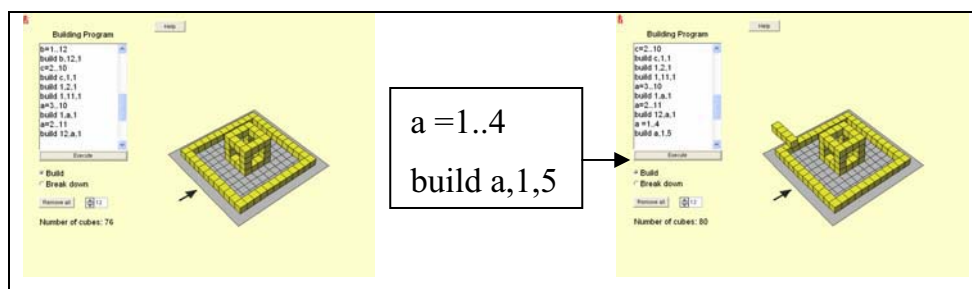


Figure 2. Students displacing the point of origin.

THE THIRD CO-ORDINATE: THE HEIGHT

In the course of the activity it was more than clear that children had extra difficulties with the third co-ordinate, that referring to height. Many times had they forgotten the third co-ordinate as it is depicted in the following excerpt.

- 1387 S2: Should we put 1,2;
 1388 S1: It doesn't fit. (they press 'execute' but the block that they want is not built)
 1389 R: There, why do you think it is not placed there?,
 1390 S1: The height is missing, the height..
 1391 S2: Oh, the height.

Moreover we observed that both teams had extra difficulties and were obliged to break and rebuild their construction many times when they were trying to place

blocks or rows of block in height other than one, that is to say not in the base level (figure 3). In the following panel it is apparent that students have built the base level of their construction just with one line of order while in the next levels cubes are placed one by one.

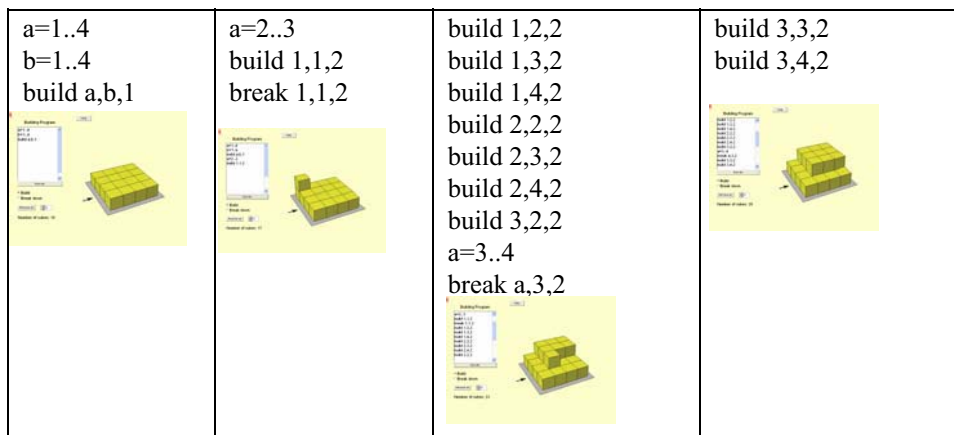


Figure 3. Students' difficulties in placing blocks in height 2.

It should be also noted that there was not a single case where a construction was made based in set(s) of perpendicular columns while children created always - either using variables or not - their constructions in successive horizontal levels, as it is obvious in the following figure.

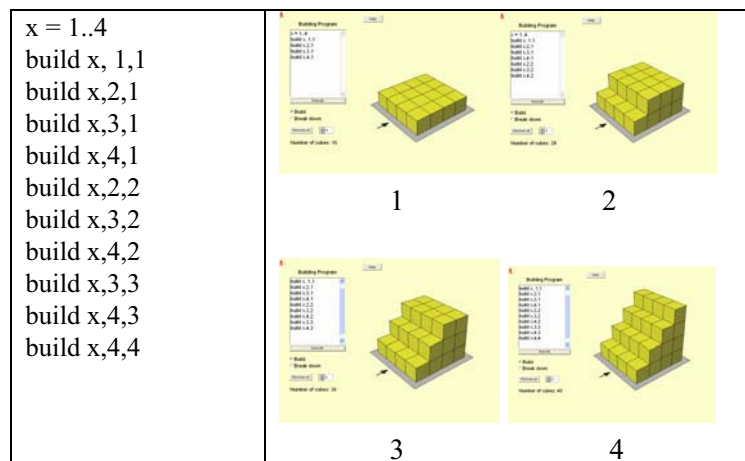


Figure 4. Students construct a building in horizontal level.

Apart from the fact that children were more accustomed to 2D representations of geometrical figures, we could point out that children resorted to this construction strategy due to the fact that they had got a 3D representation in a two-dimensional medium, the computer screen. Three-dimensional representations on the computer screen are quite new and possibly students are not yet accustomed to the conventions used so as to represent a 3D object in a 2D form to look 3D in nature (Lowrie, 2002). On the other hand if we accept the view of Dalgarno & al. (2002) that we understand

3D models through multiple 2D representations, maybe students had focused subconsciously on a simplified 2D way of visualising their constructions.

Although construction in one level is a meaningful choice as it minimizes the variables involved (children change only the numbers of 2 co-ordinates and keep stable the third one), it is still to be explained why children preferred to construct their ‘buildings’ only in the horizontal (x, y) level. In fact, this finding comes in contrast to other researches (see Dickson, Brown & Gibson, 1984) which postulate that spatial relationships are initially explored along the vertical axis and then horizontally orientated relationships are developed. In addition another research conducted by Kynigos & Latsi (2005) in relation to 3D representation of vectors in computer screen have also shown that children have focused in a simplified 2D representation but their difficulties in appreciating the third dimension were related to the appreciation of depth (y coordinate) and not of height (z coordinate).

However the differences of our results in comparison to analogous previously conducted researches, as far as spatial relationships and 3d representations are concerned, cannot be explained independently from the tool set with which students construction strategies are expressed. The affordances and conventions used by the particular 3D representation of the mathematical game ‘building blocks’ enhance and make more visually recognisable –and thus manipulable- the x, y co-ordinates through the use of a square pad as a basis for each construction. In contrast, the z co-ordinate is more obscure, there are not visual hints or perceptual cues available (for instance a cubic grid) so as to help students easily estimate the height of their constructions.

USING VARIABLES

The use of variables in order to define each coordinate was another functionality of the computational tool at students’ disposal that offered them flexibility and introduced mathematical formalism in the way of specifying and controlling each cube’s position. As it was expectable, students had difficulty in using variables, as they were not accustomed to this notion either in their mathematics or in their ICT lessons. Initially students thought that once a variable was used to represent a certain continuum of numbers, the correspondence between the letter and the set of number couldn’t be changed. The letters where conceived more as a way of naming or labelling a string of numbers rather than abstract symbols. This quasi- abstract way of conceptualising variables used in ‘building blocks’ environment is rather closely related to the notion of ‘unknown x’ where a letter, usually x, is used in the place of one and only number which is unknown but can be found if an equation is solved, a notion that is introduced in Greece at the end of primary school. The following word episode is indicative of students’ puzzlement in relation to the use of a different letter so as to represent the same set of numbers.

S2: Can we write $a = 2$ to 3; As we say $b = 2$ to 3;

R: You could have used another letter and put any numbers you think you need to have these cubes.

S1: I am not sure about what I am writing here ...

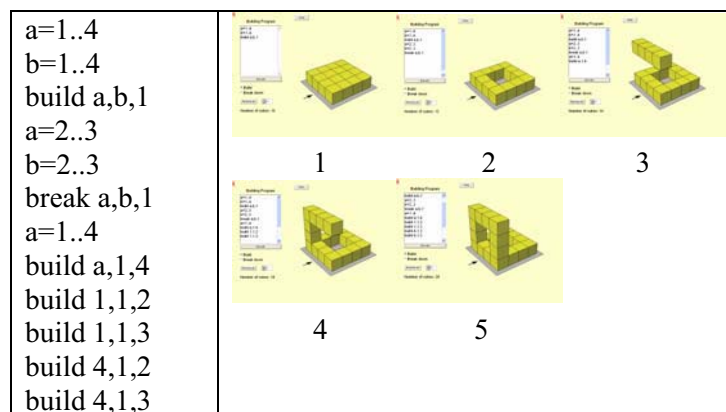


Figure 5: Use of variables and of analytic way of construction

Moreover students avoided using the same letter simultaneously to define more than one co-ordinates a time, as it is shown in figure 5 where they used different variables, a and b, for the x and y coordinates respectively, although both variables represented the same set of numbers. According to our point of view this is another indication of the transitional quasi-abstract students' conceptions of variables.

When children used variables, this was especially when creating the first level of their construction. When they were creating levels from level 1 and up, the difficulties in visualising and estimating the triplet of numbers for each column of blocks, made students resort to a more analytic way of construction, positioning the cubes one by one (figure 5).

Another interesting issue was that students in the beginning used one variable a time but gradually and especially at the end of the activities managed to use two variables simultaneously and at levels other than the base one (see figure 6). However not even once a variable was used for the third co-ordinate, that of height, a finding that underlines the difficulty that children faced in visually manipulating and experimenting with this dimension of their constructions. It follows that the use of variables was interwoven with children's way of conceiving 3d representation of space in the buildings blocks environment.

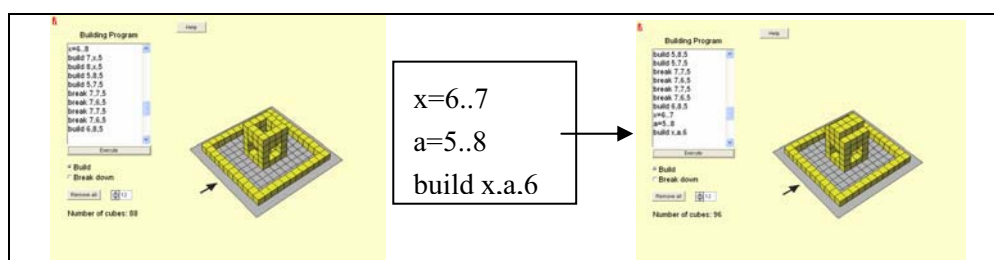


Figure 6. At the last task students used effectively two variables (x and a) at the sixth level of height.

DISCUSSION

It seems that in the computer environment of ‘building blocks’, 3d cubes acted as external objects which mediated and concretized the mathematical notion of absolute coordinate frame of reference. As a result coordinate concepts were made accessible to children at the end of primary school, even though they were not taught in the standard abstract way and with a terminology strictly related to it.

The combination of 3D representations with the use of variables and of simple everyday language commands provided a rich opportunity for students’ engagement with meaningful formalism and abstraction. What was manipulated was not just the figure itself but the 3D co-ordinates of each block. In this way accomplishing a task, e.g. constructing a building, was interwoven with control and conceptual analysis of that action. The distinction between the conceptual parts of absolute 3D coordinate frame of reference and the figural one was thus blurred unlike what is usually the case in geometrical tasks (Mariotti, 2001).

Although 3D representations in digital environments can enrich the visualisation process involved in any geometrical activity, it was more than clear in all the stages of construction that students had difficulties with the third co-ordinate that of height, a finding which can be explained not only by the fact that children are not accustomed to the conventions used to represent a 3D object in a 2D medium but also by the special representational cues of the software. However it was of great interest to us the fact that children managed to use gradually a 3d coordinate frame of reference, which implied the use of a stationary point of reference, a sense of direction for measuring each coordinate and a fixed order of coordinates (x, y, z) so as to specify each cube’s position. However, at least in the beginning, students seemed to oscillate between the fixed coordinate system and a more arbitrary egocentric one, a fact which was more prominent in cases that students were seeing their constructions from views other than the front one or when their constructions were more complicated. In this framework the use of variables was inextricably linked to students’ conceptualisation of the available 3d spatial representation depicting in an elaborate way the gradual progress as well as the limits of students’ conception of coordinates.

It goes without saying that further research is needed on the way young children engage in highly visual learning environment, as the one we used. The new kind of representations as well as the new kind of access to static representations offered by digital media, as the building blocks environment, could possibly change our conceptions about what can be learnable, in what way and at which age-group. Current curricula have been developed in the base of static and fixed systems of representations, that couldn’t be easily related one another. Our findings are possibly an indication of the need to reconceptualise curricula with the integration of technology rather than trying to fit new technologies to existing curricula.

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REFERENCES

- Cobb P., Confrey J., DiSessa A., Lehrer R., and Schauble L. (2003). Design Experiments in Educational Research. *Educational Researcher*, Vol. 32-1, 9-13.
- Dalgarno, B., Hedberg, J. & Harper B., (2002). The Contribution of 3D Environments to Conceptual Understanding, in *Proceedings of ASCILITE 2002 Conference*. pp 44-54, Auckland, New Zealand
- Dickson, L., Brown, M., & Gibson, O. (1984). *Children learning mathematics: A teacher's guide to recent research*, London, Cassel Educational Ltd.
- Goetz, J. P. and LeCompte, M. D. (1984) *Ethnography and Qualitative Design in Educational Research*, Academic Press, London.
- Harel, I. and Papert, S. (eds). (1991) *Constructionism: Research Reports and Essays*, Ablex Publishing Corporation, Norwood, New Jersey.
- Jonker., V. and van Galen., F. (2004). KidsKount. Mathematics games for realistic mathematics education in primary school, Paper presented at 10th International Conference on Mathematics Education (ICME), Copenhagen, Denmark.
- Kynigos C. (2002). Generating Cultures for Mathematical Microworld Development in a Multi-Organisational Context. *Journal of Educational Computing Research*, Baywood Publishing Co. Inc. (1 and 2), 183-209
- Kynigos, C. & Latsi, M. (2006) Vectors in use in a 3d Juggling Game Simulation, *International Journal for Technology in Mathematics Education*, Volume 13, No 1.
- Lawler R W (1985) *Computer Experience and Cognitive Development. A Child's Learning in a Computer Culture*. Ellis Horwood.
- Lowrie, T. (2002). The influence of visual and spatial reasoning in interpreting simulated 3D worlds. *International Journal of Computers for Mathematical Learning*, 7, 301-318
- Mariotti, M.A. (2001). Introduction to proof: the mediation of a dynamic software environment. *Educational Studies in Mathematics*, 44, 25-53.
- Piaget, J., & Inhelder, B. (1956). *The child's conception of space*. London: Routledge & K. Paul.

CHANGE IN PERCEPTION OF PROSPECTIVE TEACHERS REGARDING THE IMAGE OF THE TEACHER AS A RESULT OF ENGAGEMENT IN A COMPUTERIZED ENVIRONMENT

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Abstract

Calls for implementing a reform in mathematics education inspired the creation of new roles attributed to teachers. It is the teacher educators' responsibility to provide students with proper learning experiences. Such experiences should support the development of the students' abilities and skills and to enable them to adjust their teaching to the spirit of the reform. The present study aims at examining the change in prospective teachers' views regarding the image of the teacher as a result of experiencing a computerized project-based learning approach. In order to achieve this aim, we analyzed data received from two open questionnaires and written portfolios. The analysis revealed that experiencing learning in a computer-based environment has the potential to support the professional growth of prospective teachers in terms of developing their ability to reexamine the teacher's roles.

Introduction

In recent years, mathematics teacher educators have been calling to implement a reform in mathematics education (e.g. NCTM standards, 2000). The NCTM standards had influenced the development of a vision regarding the image of the new generation of teachers, and suggested various guidelines and ideas for qualifying them. In designing teacher training programs, teacher educators should consider the fact that prospective teachers (PT) begin their math methods course with interconnected ideas about mathematics, about teaching and learning mathematics, and about schools (Ball, 1988). These ideas are originated in their past experience as school students, and will eventually shape and affect their practice as mathematics teachers in the future (Skott, 2001). Most PT were taught school mathematics during the 1990s when reform documents were only beginning to affect the curriculum, resulting in having only a modicum of learning experiences that captured the spirit and vision of the standards (Lee, 2005). Consequently, PT's perceptions regarding the image of the teacher are not compatible with the image as portrayed by the standards. It is the teacher educators' responsibility to support their PT in developing insight as regards to the meaning of teaching and learning mathematics in various settings. For that matter teacher training

programs should include opportunities for experiencing learning within innovative environments, hoping that these experiences will shape the PT future teaching.

One of the focal points of the standards relates to the incorporation of technology into schools and teaching: “*Technology is essential in teaching and learning mathematics*”, since “*it influences the mathematics that is taught and enhances students’ learning*”. Based on our experience (Lavy & Shriki, 2003) we find Computerized-Project-Based-Learning (CPBL) approach as one that enables to provide opportunities to explore mathematical ideas as well as gaining insights regarding the image of the teacher, according to the spirit of the standards.

In the present study we aim at exploring the effects of learning via Computerized Project-Based-Learning (CPBL) on the change in perception of third year prospective mathematics teachers regarding the role of the teacher in a technological setting. In order to achieve this aim, an integrated assessment tool was employed: two open questionnaires, transcripts of classroom discussions, written portfolios and informal interviews (Lavy & Shriki, 2004).

Conceptual Framework and Background

In what to be followed, a brief theoretical background regarding the image of the teacher and project-based-learning is presented.

Image of a teacher. Reichel & Arnon (2005) summarized studies on PT and in-service teachers’ perceptions regarding the image of the teacher. They found that the most frequent characteristics of professional knowledge of a teacher were attributes such as: knows the subject matter well, explains patiently and clearly, teaches in an interesting manner, and instructs the students how to solve problems. Reichel & Arnon also analyzed 55 questionnaires of PT in their last year of studying, and 34 questionnaires of experienced teachers who were completing their educational studies in order to get a first degree. They were all asked to refer to the image of a good teacher. The researchers found that while most of the PT responses emphasized characteristics that refer to the personality of the teacher, most in-service teachers emphasized characteristics that refer to the professional knowledge of the teacher.

Wilson, Cooney and Stinson (2005) examined the perspectives of nine high school teachers as regards to ‘good mathematics teaching’, using three interviews. They found that all their interviewees “*emphasize the importance of prerequisite teacher knowledge, promoting mathematical understanding, engaging students, and effectively managing the classroom environment*” (p. 91). As to the prerequisite knowledge, they reported that teachers should have knowledge of mathematics and their students. Some of their interviewees believe that what constitutes good teaching is making connections in mathematics, helping students to visualize mathematics, assessing students’

understanding, and simultaneously considering various aspects concerned with class management.

As regard to teaching in technology environments, Farrell (1996) identified six types of roles that teachers assume: (1) *a manager* of the classroom; (2) *the task setter*, who ask questions but decides most strategies for doing mathematics; (3) *explainer* who gives rules and sets the focus of any problem solving; (4) *a counselor* who advises and encourages and stimulates students problem solving; (5) *a fellow investigator* who participate in problem solving along side students; (6) *a resource of information* both mathematical and technical.

The teacher's image in Reichel & Arnon (2005) and Wilson et. al., (2005) is viewed as one who has to explain well and promote mathematical knowledge. However, there are no indications of appropriate guidelines aimed at achieving those goals. Whereas the image of the teacher when teaching in technological environments, like in Farrell's study (1996), is characterized more specifically, and emphasize the teacher as a counselor and a fellow investigator. Realizing the importance of integrating technology into the mathematics classroom and since the ways individual teachers take up using technology in their mathematics classrooms is usually consistent with their formal teaching practices (Kendal and Stacey, 2001), as teacher educators we need to incorporate the use of technology in the teacher training programs in order to enhance the desired change in perceiving the essence of teaching among the future teachers.

Project-Based-Learning. PBL is a teaching and learning strategy that involves students in complex activities, and enables them to engage in exploring important and meaningful questions through a process of investigation and collaboration (Krajcik, Czerniak and Berger, 1999). The PBL approach allows students to pose problems, ask questions, make predictions and decisions, design investigations, collect and analyze data, use technology, share ideas, build their own knowledge by active learning, and so on. The approach is based on the idea that students should work relatively autonomously over a long period of time and conclude their work with products or presentations (Thomas, Mergendoller and Michaelson, 1999). The PBL approach has various advantages (e.g. Krajcik, Blumenfeld, Marx, Bass, Fredricks and Soloway, 1998): it develops a sense of personal contribution to the process of learning; increases motivation; raises self satisfaction; helps in developing long-term learning skills and a deep, integrated understanding of content and process; increases the ability to share ideas; promotes responsibility and independent learning; provides answers to different learning needs; develops the ability to collect and present data, and so forth.

In our study, the PT used dynamic geometry computer software, which served as a fruitful environment for posing problems, making experiments, observing stability/instability of phenomena, stating and verifying or refuting conjectures easily and quickly (Marrades & Gutierrez, 2000). Since the computerized environment was an

integral part of PBL approach, we termed it as Computerized-Project-Based-Learning (CPBL) (Lavy & Shriki, 2003).

Students might also encounter several difficulties while learning through the PBL approach. Among them: inability to generate meaningful questions; troubles in managing complexity and time; problems in processing data and developing a logical argument to support claims (Krajcik et al., 1998). As a consequence it is recommended to incorporate some "scaffolds" within the PBL process in order to help the students overcome their difficulties.

By implementing the CPBL approach we intended to help the PT learn and discuss the concepts of the discipline, either educational or mathematical, and enable them to experience the meaning of 'innovative learning approach'. We were hoping that such an experience would result in a change in their perception regarding the image of the teacher.

The study

The study participants. 25 PT (8 male and 17 female) from an academic college, in their third year of studying towards a B.A. degree in mathematics education participated in the course. The students represented all talent levels. The students are graduated towards being teachers of mathematics and computer science or teachers of mathematics and physics in high school.

The course. The course, in which the research was carried out, is a two-semester course and is part of the students' teacher training in mathematics education. At the first semester, these approaches are demonstrated via geometrical topics while at the second semester via algebraic topics. The research was carried out in the first semester. The project was geometry-oriented, and the students used dynamic geometrical software in the various stages of the project. The project included the following phases: 1. Solving a given geometrical problem that served as a starting point for the project; 2. Using the 'what if not?' strategy (Brown & Walter, 1993), to create a range of new problem situations on the basis of the given problem; 3. Choosing one of the new problem situations and posing as many relevant questions as possible; 4. Concentrating on one of the posed questions and looking for suitable strategies for solving it; 5. Raising assumptions and verifying/refuting them; 6. Generalizing the findings and drawing conclusions; 7. Repeating stages 3-6, up to the point in which the student decided that the project had been exhausted. The PT were exposed to three modes of interactions: (i) Team work; (ii) Whole class; and (iii) Student-teacher interaction.

The **team work interaction** included the cooperative work of two PT in order to accomplish the project, and the preparations for presenting the work during the class sessions. **The whole class interaction** included presentations of the projects and monitoring a classroom discussion afterwards. Throughout the discussions the PT talked

about their difficulties and asked for their classmates' advice. The classroom discussions that followed the presentations provided the PT with a supportive environment while working on the project. During the class discussions we helped the presenters to focus their major ideas, monitor the discussion, and understand the essence of their classmates' suggestions and questions. **The student-teacher interaction** included a continuous exchange of e-mails. The PT were asked to handle a *portfolio* on a regular basis. As was mentioned, the portfolio served as one of the evaluative tools for assessing the change in views of the PT regarding their image of the teacher.

Data Collection and analysis. In the present study we analyzed the PT's portfolios and the questionnaires. Analysis was ongoing and continually informed the data gathering process. When the data collection phase was completed, we followed the process of analytic induction (Goetz & LeCompte, 1984), reviewing the entire corpus of data to identify themes and patterns and generate initial assertions regarding the image of the teacher in a computerized environment. We followed three consecutive phases: firstly, we scanned all the students' utterances from the questionnaires throughout the lenses that concerned their perception of the image of a teacher looking for changes between utterances from the beginning of the project and at the end of it. Secondly, we categorized the utterances according to their content. Thirdly, for each of the aforementioned category, we looked for relevant evidences in the students' portfolios.

Results

When scanning the research data, we noticed that the PT's utterances could be classified according to three main aspects: affective, cognitive and didactical. In the scope of this paper we focus only on the didactical aspect.

Table 1 presents the PT's utterances regarding didactical aspect of the teacher's role in class before and after the engagement in CPBL approach. The utterances came from a hand-full of the study subjects. The percentage in the table represents the ratio between the number of students who explicitly wrote the specific utterances and the number of the study subjects. For example, the utterance: "*Knows how to explain well*" was uttered by 22 students out of 25 (88%).

Frequency (percentage)	The PT's Utterances	
22 (88%)	1. Knows how to explain well	Before
16 (64%)	2. Teaches in an interesting manner	
11 (44%)	3. Tries various teaching methods in order to find the optimal one	
9 (36%)	4. Uses many examples and games	
8 (32%)	5. Adjusts the learning materials to the students' level	

Working Group 9

21 (84%)	1. Encourages her students to ask questions	After
21(84%)	2. Provides her students the opportunity to explore mathematics, and does not give them the solutions right away	
20 (80%)	3. Motivates her students to experience the process that a mathematician goes through while looking for mathematics regularity	
19 (76%)	4. Develops creative thinking	
19 (76%)	5. Uses many class discussions	
17 (68%)	6. Explains each topic clearly and simply	
17 (68%)	7. Improves the mathematical thinking	
17 (68%)	8. Inspires the students to look for a deep understanding and not just success	
16 (64%)	9. Enables all students to take an active part in the mathematics lessons	
14 (64%)	10. Uses many examples in class for clarification	
14 (64%)	11. Lets the students think first and then establishes the subject matter	
15 (60%)	12. Teaches attractively	
13 (52%)	13. Integrates interesting activities in his teaching	
11 (44%)	14. Manages to understand the students' difficulties	

Table 1: the PT's utterances regarding the teacher's functioning in class

Examination of the PT's utterances before and after the work in a computerized setting reveals a change in their views regarding the role of the teacher. The change comes into fruition both in the *quantity* and the *content* of the utterances. As regards to the quantity, before the work on the project there were only 66 utterances while at the end of the semester there were 234 utterances. The number of utterances after completing the project more than tripled itself also due to the variety of the utterances (14 vs. 5). This implies on the PT ability to encompass more aspects regarding teaching.

Before. The PT's utterances before the implementation of the CPBL approach related to the **teacher as a mediator between her class and the subject matter and general means for achieving educational goals**. As to the former, the PT specified various ways the teacher should mediate between her students and the subject matter. The PT pointed out a general purpose: *Teaches in an interesting manner* and various abilities for fulfilling this purpose such as *knows to explain well; Tries various teaching methods in order to find the optimal one; Uses many examples and games and adjusts the learning materials to the students' level*. These utterances concerning the didactical skills of the teacher are general in their essence. Namely, they do not mention any specific learning/teaching method; they do not specify any kind of games and examples. The PT suggest that the teacher should *try various teaching methods in order to find the optimal one* but do not really indicate any specific method. In addition they believe that the

teacher should *know to explain well*, but they do not mention how or what exactly they mean by that. The following utterances: *Tries various teaching methods in order to find the optimal one; Uses many examples and games; adjusts the learning materials to the students' level* refer to **general means for achieving educational goals**, without any indication of the goals themselves. Moreover, the PT did not mention any connections between the teacher's didactical skills and their implications to students' learning and understanding.

From the PT's utterances we can conclude that at this stage their views regarding the teacher's role is rather basic and superficial.

After. Analysis of the utterances after the experiencing of the CPBL approach reveals that the PT related to three main categories: **the teacher as a mediator between her class and the subject matter; educational goals and means for achieving them, and teacher-student interaction**. As to the category: the teacher as a mediator between her class and the subject matter there were utterances such as: *Explains each topic clearly and simply; teaches attractively*. With regard to the category: educational goals and means for achieving them, there were utterances such as: *Develops creative thinking; Improves the mathematical thinking*, which relate to educational goals while the utterances indicate means for achieving educational goals. As to the category: the teacher-student interaction, there were utterances such as: *Manages to understand the students' difficulties; Encourages her students to ask questions*.

After the implementation of the CPBL the teacher has both educational goals and the means for achieving educational goals. The main educational goals concern developing students' thinking, and directing them while looking for mathematical regularities. The means include: providing the opportunity to explore and pose questions, and create an environment that would enable the students to discuss their problems and questions.

Discussion

In this section we discuss the nature of change in viewing the teacher's image by our PT after their engagement in learning via CPBL approach. We also refer to the effect of the CPBL approach on the observed change.

The nature of change in viewing the image of the teacher. In general, it can be said that the change in the PT's views come to fruition in modification of the nature and scope of the utterances. After experiencing the CPBL, the utterances became more comprehensive on one hand and more detailed on the other. The utterances included a larger range of attributes the teacher should have than the ones that were uttered at the beginning of the semester. Analysis of the utterances in Table 1 reveals an interesting phenomenon. All the utterances *before* the implementation of the CPBL approach involve the teacher's centrality. Namely, the PT could see only themselves and consider merely issues that related to the teacher's role, though this teacher is described as a

'vague' figure, with undefined educational goals. However, the utterances *after* the implementation of the CPBL approach related also to the class students. The PT began to internalize the fact that teaching is not simply a teacher-centered activity but it is affected by the presence of students. That is, teachers should adjust their teaching according to feedback coming from their students. This finding is consistent with some of teacher's roles described in Farrell's findings (1986) mainly the teacher as a task setter, as a counselor and as a fellow investigator. The description of that teacher specifies educational goals, and a detailed list of means aimed at achieving them. Yet, the PT did not discuss any issue concerning teacher–students interaction. The changing nature of statements made by the participants indicated that they had gained insights into the complexity of teaching even before beginning their work inside high school classrooms.

The effect of the CPBL approach. In applying the CPBL we intended to engage our PT in a teaching and learning environment which would enable them to explore important and meaningful questions through a process of investigation and collaboration (Krajcik, et al., 1999). We believe that such an experience would provide them with a sense of personal contribution to the process of learning, increases their motivation, increase the ability to share ideas, develop their ability to collect and present data, and so forth (Krajcik et al, 1998). In addition, we were hoping that the PT's experience with this approach would improve their mathematical knowledge, and expose them to the benefits of this method as a means for teaching and learning mathematics in the spirit of the NCTM standards (2000). Observation of Table 1 demonstrates a list of utterances referring to constituents of the CPBL approach, such as: *Encourages her students to ask questions, provides her students the opportunity to explore mathematics, and does not give them the solutions right away, develops creative thinking, integrates interesting activities in his teaching, uses many class discussions*, such utterances did not appear before the engagement in the CPBL approach. Analyzing the PT's portfolios we realized that the CPBL approach had a crucial effect on changing their perceptions as regard to the image of the teacher. The analysis took into consideration the distinction between the components of the learning environment: the work with the computer software, the class discussion, the interaction between the students and the interaction between the students and us. In the scope of this paper we briefly relate to **the effect of the work with the software**. Many PT reflected in their portfolios on ideas similar to the following: *“The work with the software was the enjoyable part of the project. I can raise any idea and immediately examine it. Without the software it was impossible since it requires a great deal of technical work... the software also gives the impression that no matter what course of inquiry you choose – you will always reach an idea that will eventually end in some mathematical discovery”*; *“The software enables me to express my wildest ideas. I would have never dared to do it before. Here I am not afraid of my*

class mates' reactions and I can make mistakes. I am sure that only if you dare you can eventually discover something. I have never thought this way before”.

As can be seen, the PT related to various aspects that concern the work with the software: the software as a facilitator which enables the learner to focus on looking for inquiry courses, instead of wasting time on technical work; the software **as a mediator**, due to its **interactive nature**, which allows the learner to receive an immediate response to his trials. Consequently, the learners can assign themselves to the process thinking and learning. Moreover, the work with the software made the PT realize that learning situations have an effect on the students learning: *“In school we were used to the situation in which the teacher provided us problems from textbooks and we had to solve them. It is obvious that when you are asked to prove a theorem it means that the theorem is valid. Here [in the project] it is different. I don't know what I am looking for, and what I will eventually find. I have both to formulate the regularity and then prove it. It is much more intriguing and challenging but can also end up with disappointment and frustration”.* In this excerpt the PT refers to a specific characteristic of the work via the software - the ability to examine the validity of any conjecture. It is important to note that this characteristic stems from the features of the software which we used.

Concluding remark

Various studies focus on the influence of learning within a computerized environment on the development of mathematical knowledge and understandings. In this paper we demonstrate a case in which experiencing learning in a computer based environment has the potential to support the professional growth of PT in terms of developing their ability to reexamine the teacher's roles and gaining new insights regarding the complexity of teaching even before the begin their actual practice. The students' reflections support the fact that the interactive nature of the software was the most influential characteristic among all others in developing the image of the teacher and her roles. The interactive nature of the software enabled the PT to: (a) pose problems and easily translate them into inquiry tasks; (b) look for regularities; (c) test conjectures and arrived at conclusions. The interactivity also enabled them to work fluently and freely without being afraid of criticism or judgment. To conclude, the computerized environment provided the PT a supportive environment and consequently changed their perceptions as regards to the meaning of teaching and learning.

References

- Ball, D. (1988). Unlearning to teach mathematics, *For the Learning of Mathematics*, 8(1), 40-48.
- Brown, S. I. & Walter, M. I. (1993). Problem Posing in Mathematics Education. In Stephen I. Brown & Marion I. Walter (Eds.) *Problem Posing: Reflection and Applications*, Hillsdale, New Jersey: Lawrence Erlbaum Associates. 16-27

- Farrell, A. (1996). Roles and behaviors in technology – integrated precalculus classrooms. *Journal of Mathematical Behavior*, 15, 35-53
- Goetz, J.P. and LeCompte, M.D. (1984). *Ethnography and Qualitative Design in Educational Research*. Academic Press.
- Kendal, M. and Stacey, K. (2001). The impact of teacher privileging on learning differentiation with technology. *International Journal of Computers for Mathematical Learning* 6(2), 143–165.
- Krajcik, J. S., Blumenfeld, P. C., Marx, R. W., Bass, K. M., Fredricks, J., & Soloway, E. (1998). Inquiry in project-based science classrooms: Initial attempts by middle school students. *The Journal of the Learning Sciences*, 7, 313-350.
- Krajcik, J., Czerniak, C. & Berger, C. (1999). *Teaching Children Science - A Project-Based Approach*, McGraw-Hill College, New York
- Lavy I. & Shriki, A. (2003). Pre-service teachers' transition from "knowing that" to "knowing why" via computerized project-based-learning. In N. A. Pateman, B. J. Dougherty and J. Zilliox (Eds.), *Proceedings of the 27th International Conference on the Psychology of Mathematics Education*, Vol. III (181-187). Honolulu, Hawaii.
- Lavy I. & Shriki, A. (2004). An integrated approach for assessing pre-service teachers' professional growth – the case of portfolio and class discussion, paper presented in the 10th International Congress in Mathematics Education, http://www.icme-organisers.dk/tsg27/papers/02_Lavy-Shrik_fullpaper.pdf
- Lee, H. S. (2005). Facilitating students' problem solving in a technological context: prospective teachers' learning trajectory. *Journal of Mathematics Teacher Education* 8, 223-254
- Marrades, R. & Gutierrez. A. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics*, 44, 87-125.
- NCTM - National Council of Teachers of Mathematics (2000)., *Principles and standards for school mathematics*. Reston, VA: NCTM.
- Reichel N. & Arnon, S. (2005). Three portraits of teachers in the eyes of the teaching student: the ideal teacher, the teacher educator and the student's image of himself as a learner. *Dapim*, 23-58 (in Hebrew)
- Skott, J. (2001). The emerging practices of novice teachers: The roles of his school mathematics images. *Journal of Mathematics Teacher Education*, 4(1), pp. 3-28.
- Thomas, J. W., Mergendoller, J. R., and Michaelson, A. (1999). *Project-based learning: A handbook for middle and high school teachers*. Novato, CA: The Buck Institute for Education.
- Wilson, P.S., Cooney, T. J. and Stinson D.W. (2005). What constitutes good mathematics teaching and how it develops: nine high school teachers' perspectives. *Journal of Mathematics Teacher Education*, 8, 83-111

MATHEMATICS LEARNING WITH THE USE OF THE BALANCE, A COMPUTER PROGRAMME FROM ENCICLOMEDIA

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This study has as its main purpose the analysis of primary school students' mathematical activity in a sequence of lessons on operations with fractions and equivalence of fractions. In these lessons, an interactive programme called The Balance designed for Enciclopedia, a national project in Mexico, was used. Using enactivism as theoretical framework the researchers observed the lessons, discussed with the teachers and talked to students. From the analysis of the observations it was found that The Balance was an instrument which invited students to act mathematically in a number of ways, and helped them change their actions from trial and error to the use of systematic procedures such as arithmetic operations and comparison of fractions using different representations.

INTRODUCTION

The introduction of digital technologies in the teaching of mathematics has been considered by some as an answer to the problems students present when they learn mathematics, allowing for the development of conceptual understanding. This is not always the case, as tools often introduce different problems and their use generates new sets of questions about student's learning (Lagrange, et. al., 2001; Laborde, 2004). In this paper, we investigate and analyse the relationships between the use of a computer programme, specifically designed for a large-scale Mexican project called Enciclopedia, and students' mathematical learning.

Enciclopedia was devised with the purpose of enriching primary school teaching and learning by working with computers in the classrooms. An electronic version of the mandatory textbooks that are used in all primary schools in Mexico is being enhanced with links to computer tools designed to help teachers with the teaching of all subjects. As members of the Mathematics group in Enciclopedia, we create resources and strategies which can help teachers and students in their teaching and learning of mathematical concepts. An additional part of our work is to investigate how students learn mathematics as they use the computer tools that Enciclopedia provides them with.

To begin with, we consider some theoretical ideas about the learning of mathematics; in particular, with the use of computer interactive programmes. We also give a brief description of the interactive programme that is being used in mathematics lessons and that we want to consider in our analysis. We discuss the way in which the learning of mathematics with these programmes is being investigated. We talk about

the approach we have taken in Enciclomedia and the methods we are developing for our project, **and discuss some of the results obtained.**

THEORETICAL FRAMEWORK

Our ideas about learning are based on enactivism, a theory of knowing which considers learning as effective or adequate action (Maturana and Varela, 1992). In enactivism, our minds are seen as ‘embodied’ and cognition as ‘embodied action’. These ideas of ‘embodiment’ entail two fundamental senses: on the one hand cognition is seen as ‘dependent upon the kinds of experience that come from having a body with various sensorimotor capacities’ and on the other, individual sensorimotor capacities are considered to be ‘themselves embedded in a more encompassing biological and cultural context’ (Varela, 1999, p. 12). The first meaning of embodiment locates cognition in our bodies, and prevents us from thinking about it as an abstract notion that is detached from our everyday experience. The second situates learning in a wider social and cultural context.

In enactivism learning occurs when, as a result of interactions with each other and with the world, individuals act in a way that is effective in a certain domain. When faced with a problematic situation, individuals act by making use of what they already know and are prepared to do within a certain context, according to their biological and social history. Their actions can be effective in some situations and ineffective in others.

In a particular context or location, the participants create together the conditions that allow actions to be adequate or effective. Learning outcomes cannot be predetermined or predicted, but the criteria for the adequateness of actions are, at least in part, specified by teachers and students. In mathematics classrooms, certain kinds of behaviour are promoted while others are discouraged. For example, in a particular classroom, following specific procedures and giving right answers might be an expected behaviour, while in others, exploring and looking for multiple ways of solving a problem might be common activities. In a given context, the behaviour of all the participants will be modified in a similar way, according to the conditions that make actions effective in that environment.

Learning mathematics with computer tools

From an enactivist perspective, the use of computer tools is part of human living experience since ‘such technologies are entwined in the practices used by humans to represent and negotiate cultural experience’ (Davis et. al., 2000, p. 170). Tools, as material devices and/or symbolic systems, are considered to be mediators of human activity. They constitute an important part of learning, because their use shapes the processes of knowledge construction and of conceptualization (Rabardel, 1999). When tools are incorporated into students’ activities they become instruments, which are mixed entities that include both tools and the ways these are used. They are not

merely auxiliary components in the teaching of mathematics; they shape students' actions and therefore their learning (ibid, 1999).

Every tool generates a space for action, and at the same time it poses on users certain restrictions. This makes possible the emergence of new kinds of actions. The influence that tools exercise on learning is not immediate. Actions are shaped gradually, in a complex process of interaction. In the classrooms, students construct meanings through the use computer tools, in a process of social interaction and with the guide of the teacher (Mariotti, 2001).

The purpose in Enciclomedia is to develop initiatives that can help teachers to create contexts in which certain actions, related to the learning of different mathematical concepts, can be fostered through the use of computer tools. Because we believe learning occurs in the process of interactions, when students gradually modify their behaviour, we create digital resources and teaching guides that promote the joint exploration of mathematical ideas and concepts. Computer programmes in Enciclomedia are intended to broaden users' experiences with mathematics by providing spaces where the need for using mathematical procedures arises naturally. Our activities are meant to challenge and develop students' (and teachers') intuitive (and sometimes inadequate) thinking and to trigger, in them, ways of acting that we consider mathematical.

RESEARCH QUESTIONS

We are interested in investigating those mathematical activities that we find to be effective as students work in mathematics problems using *Enciclomedia*. In particular, in this study, we address students' mathematical learning with the use of a computer interactive programme called *The Balance*. This is the second part of a previously reported investigation in which we analysed how three teachers used the programme to teach fractions (García and Trigueros, 2005). Here we address the questions: 'How does the use of *The Balance* contribute in shaping students' learning?' 'What are the mathematical actions of students during lessons where the interactive programme is being used?'

ANTECEDENTS

The computer programme *The Balance* has been devised with the purpose of helping teachers and students in the teaching and learning of rational numbers. A general overview of the research literature in this area indicates that although there is agreement on the inherent complexities of the teaching and learning of rational numbers (Hunting and Davis, 1997; Mack, 1998; Hecht, 1998; Hunting et al., 1998; Moss and Case, 1999; Tzur, 1999; Cramer et al., 2002; Litwiller, 2002), there is no consensus about how to facilitate the understanding of the concepts related to these numbers and their operations (Behr, et al, 1997; Taube, 1997). A great number of research reports are focused on identifying the experiences children need to develop

meanings for rational numbers (Taube, 1997) and several research studies have built upon students' and teachers' difficulties with concept of fraction (Davis and Thipkong, 1991; Behr et al, 1997; Cramer et al., 2002). For example, The Rational Number Project Curriculum (Cramer et al., 1997, Cramer et al., 2002), includes activities to work with rational numbers that emphasize translations within modes of representation. Results showed that when these materials were used, students approached mathematical tasks that involved fractions in a conceptual way. Other studies concerned with the same project concluded that children's learning about fractions can be enhanced with the use of multiple concrete models. It was also shown that students benefit from opportunities of talking to one another and to their teachers about their ideas about fractions. Finally, authors conclude that teaching materials should focus on the development of conceptual knowledge prior to formal work with symbols and algorithms (Cramer et al., 1997).

The Balance was created as a space in which students and teachers could explore their ideas about rational numbers by working with activities involving equivalent fractions. It reproduces a problem in the students' mandatory textbook in which there is a representation of a mobile toy which needs to be balanced on different levels by using fractions. The computer programme provides automatic feedback that helps users in identifying which parts of the toy are balanced and which are not. In enactivist terms, the programme is meant to trigger in students mathematical effective actions such as comparing rational numbers, and finding equivalent fractions.

In a previous study, it was shown that many teachers have difficulties with the concept of equivalence of fractions, but that the use of *The Balance* can help them to reconsider their strategies and to understand the purpose of the activities in the textbook (García and Trigueros, 2005). It was also found that *The Balance* helped teachers reflect on the concept of equivalence of fractions and that, when the programme was used in the classroom, it enhanced the use of several representation registers (*ibid*, 2005). The evidence in that study, however, was insufficient to conclude that the tool helped students in their learning of the concepts related to fractions. We address students' learning with *The Balance* more directly in this study, using an enactivist framework.

METHODOLOGY

In enactivism, research is considered to be a way of learning, and therefore researchers are seen as individuals developing their learning in a particular context. From this perspective, researchers interpret the world in a particular way, influenced by their previous experiences. In addition, in the process of doing research, researchers influence and shape the context in which they are immersed (Reid, 1996, p. 206). The interdependence of context and researchers makes the research process a flexible and dynamic one. Research does not occur in a linear fashion; rather, it is seen as a recursive process of asking questions. The work reported in this paper is part of a complex process of interaction and development of ideas. Because of the

nature of our work we consider it to be not only research but ‘action research’. The methods we use to investigate mathematics learning will change in the future according to what we observe in the classrooms and to the feedback we receive from colleagues.

In order to research the learning of mathematics with *The Balance*, we contacted a school in Mexico City where we worked with two Year 5 and two Year 6 groups of about 25 students each (aged 11-13). Two of us visited the classrooms at a time and our role was that of participant observers. We helped the teacher in giving general directions on how to use the computer programmes and we walked around the room, making comments or asking questions about students’ work. As we entered the classrooms we contributed in creating certain kinds of classroom cultures – that is, patterns of actions and interactions. When digital technologies are used, these change the way students and teachers interact with each other and therefore particular classroom cultures emerge.

In order to analyse students’ learning with *The Balance*, we carried out detailed observations of students’ actions in the classrooms. We used multiple methods for the collection of data. We used audio recording during the lessons. We recorded whole group discussions as well as interactions that occurred between two or three students and/or between students and teachers or researchers. We also used a video camera, with the purpose of recording, for each lesson the actions of particular pairs of students. So far we have videoed different students on every session.

Additionally, for each lesson, we filled in two observation sheets developed using the enactivism framework. We used the first to obtain a sense of the culture in the classroom through the following aspects of students’ effective behaviour: *Active/Passive, Attentive/Inattentive, Working with others/Working individually, Freedom/Constraint, Giving correct answers/ Formulating explanations, Understanding/Remembering*. These aspects had emerged in a previous study in which they had been helpful in analysing students’ mathematical actions (Lozano, 2004). The second observation sheet refers more specifically to mathematical behaviour, and includes the following headings: *Initial mathematical behaviour* (which refers to students’ actions related to mathematics during the whole group introductory discussion at the beginning of the lesson), *Mathematical actions* (those observed during the rest of the lesson, which are related to the mathematical concept(s) in the textbooks’ chapter) and *Other mathematical actions* (they do not explicitly address concepts in that chapter). Particular incidents, where mathematical behaviour is observed, were written at length under each heading. In addition, we have kept records of students’ work with paper and pencil.

Acting mathematically does not necessarily mean, to us, solving a problem in a conventional ‘correct’ manner. We collectively decide on what is mathematical by having discussions in which we talk about our notes, our transcripts from the audio tapes, and about what we observe on the videos. To support our interpretations about

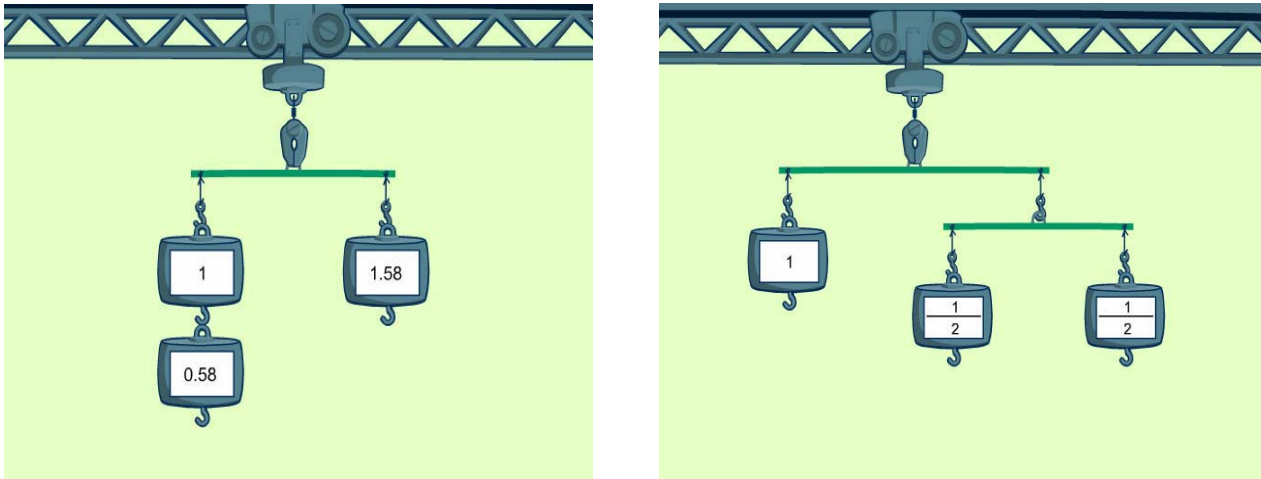
mathematical actions, we also read the literature on the teaching and learning of the concepts related to fractions, that we already mentioned as the antecedents to this study to identify the experiences and actions of students with *The Balance* that facilitate the conceptual understanding of the concepts related to fractions, and the use of different representations. We use the textbooks to identify these concepts, and to learn about the purpose of the chapters in them.

RESULTS

We observed six sessions in which students from two different school-groups worked with *The Balance*. During these sessions we identified mathematical actions at the beginning, during and at the end of every lesson. The purpose of the sequence of the lessons, as found in teachers' lesson plans was 'to solve problems by adding or subtracting fractions using the concept of equivalence of fractions'. At the beginning of the sequence of lessons, teachers conducted an exploratory discussion which included comparing fractions and decimal numbers. We observed that most students were not able to compare decimal numbers and fractions adequately. Some would think that, for example, .4 is equivalent to .04, or that the latter is greater than the former. Other students would also say that $\frac{1}{3}$ is greater than $\frac{1}{2}$ because 3 is greater than 2. During following sessions, students were asked, in whole-group discussions, about addition and subtraction of fractions. Our records indicate that some students often added numerators and denominators when adding fractions and that they applied procedures algorithmically, without being able to explain what they were doing.

After these initial explorations, *The Balance* was introduced to the groups. In the beginning, teachers closely guided students' actions but later students were left to explore and to work on their own. At first, *The Balance* was used to compare weights with pairs of numbers such as the following: 2 and 1, .04 and .4, .04 and .040, .4 and $\frac{4}{10}$. The computer programme indicates, visually and with sounds, whether the weight of two arms of the balance are equivalent or not and we repeatedly observed students having to modify their answers due to the feedback they received.

Later on, students were asked to build more complicated models and to reproduce the problems posed in their textbooks, which often involve addition, subtraction and even multiplication or division of fractions:



As students used *The Balance*, they often encountered situations in which they did not know, immediately, how to balance the mobile. In this case, it was observed that students started out by answering with trial and error. However, they gradually refined their strategies. In small groups they started discussing the use of better strategies, and they produced more efficient methods for obtaining fractions that equilibrated the balance. We observed them using different kinds of drawings and testing some hypotheses with the programme. Some of them found systematic ways of solving problems with the balance, finding out when it was appropriate to add or to subtract fractions. Using ‘random’ numbers proved to be an inadequate manner of addressing the problem. Effective behaviour included looking for systematic ways of balancing the mobile.

In every session, whole-group discussions were organised, during which students were asked to explain their strategies for balancing the scales. Explanations differed in their degree of mathematical adequacy and in the use of representations of fractions, but some students could clearly verbalize their strategies and exhibited conceptual understanding related to comparison and operations with fractions. Teachers and students, together with *The Balance*, determined the adequacy of the answers and explanations.

When the students worked with *The Balance*, mathematical actions such as the following were registered:

- Students ask questions such as ‘Why is this heavier than this?’
- Students give explanations about how to equilibrate the balance: ‘fractions have to get smaller’ ‘this and these two have to be the same; you need one that is the same here’.
- Students perform arithmetic operations with fractions and decimal numbers.
(28/11/05)

The data show that the use of *The Balance* invites students to act mathematically in a number of ways: comparison of numbers, verification of hypothesis based on the use

of other representations, use of operations with fractions. They work with mathematical concepts such as equivalent fractions and they operate on mathematical symbols. When students make mistakes, they often reconsider their answers after using *The Balance*:

Student1: You have to add this and this, see, $\frac{1}{3}$ add $\frac{1}{3}$, that is $\frac{2}{6}$ (They try their answers in *The Balance*)..... No, it doesn't work...

Student2: No, it is $\frac{2}{3}$!! $\frac{1}{3}$ add $\frac{1}{3}$ is $\frac{2}{3}$, not $\frac{2}{6}$. (06/12/05)

Different representations were used by students when dealing with fractions. While working with *The Balance* they often used paper and pencil and resorted to numerical and pictorial representations of fractions. Mathematical questions such as 'why is not $\frac{1}{3}$ heavier than $\frac{1}{2}$? Mathematical explanations were given by students within small groups and in whole-group discussions. Justifications were usually incomplete although we have also recorded sophisticated explanations, such as using graphic representations (drawings of pizzas) to show how a fraction with odd denominator could be divided in two.

CONCLUSIONS: SOME REFLECTIONS ON METHODOLOGY AND DIRECTIONS FOR FUTURE RESEARCH

Learning mathematics is a complex process. The introduction of digital technologies does not automatically generate mathematical understanding. In fact, different computer programmes trigger in students different kinds of behaviour, as we have observed during our analysis of mathematics learning with the programmes from Enciclomedia (see Lozano *et al.*, 2006).

When investigating the learning of mathematics with *The Balance*, we found that the programme invited students to act mathematically; using concepts from the textbooks in a variety of ways. We noticed that students started looking for explanations which could help them interact with their peers and teachers as they talked about their work with *The Balance*. Over time, patterns of effective behaviour were shaped and students acted mathematically; that is, in terms of an enactivist perspective, mathematical learning occurred. Students gradually modified their actions from trial an error to finding systematic methods to solve the problems which included the use of operations with fractions and comparison of fractions using equivalence of fractions. Changes were also observed in the explanations of their procedures. Explanations included several mathematical concepts and the use of different representations of fractions.

From our observations we can say that the use of *The Balance* triggered changes in students' behaviour. Feedback provided by the tool contributed in shaping students' learning. *The Balance* became an instrument in their activity, and as such, it became part of their mathematical actions and explanations. This was only possible with teachers' support and intervention. Even when *The Balance* proved to be useful as a means of enriching children's experiences with fractions, teachers' organisation and

strategies were crucial.

The theory used and methodology followed during this part of the study proved to be effective for recording, with detail, the activities and behaviours during the class. This can be considered a contribution of this study. The use of the computer programme as a tool, facilitated students' conceptualization of fractions, and promoted the use of different representations, activities that are closely linked to the learning of fractions. Students felt confident with the presence of the researchers and talked freely with them. More research is needed to study how students use what they learned while working with *The Balance* when they face other lessons or problems where operations with fractions and equivalence of fractions are needed.

REFERENCES

- Behr, M. J., Khoury, H. A., Harel, G., Post, T. and Lesh, R.: 1997, 'Conceptual units analysis of preservice elementary teachers' strategies on a rational-number-as operator task', *Journal for Research in Mathematics Education*, 28(1), 48-69.
- Cramer, K., Behr M. J., Lesh, R. and Post, P.: 1997, *The Rational Number Project Fraction Lessons: Level 1 and Level 2*. Dubuque, Iowa: Kendall/Hunt Publishing.
- Cramer, K., Post, T. R., & del Mas, R. C.: 2002, 'Initial fraction learning by fourth- and fifth-grade students: A comparison of the effects of using commercial curricula with the effects of using the Rational Number Project curriculum', *Journal for Research in Mathematics Education*, 33(2), 111-144.
- Davis, E. J. and Thipkong, S.: 1991, 'Preservice elementary teachers' misconceptions in interpreting and applying decimals', *School Science and Mathematics*, 91(3), 93-99.
- Davis, B., Sumara, D. and Luce-Kapler, R.: 2000, *Engaging Minds: Learning and Teaching in a Complex World*. London: Lawrence Erlbaum.
- García Tello, H. and Trigueros, M.: 2005, 'The Balance, an interactive activity to work with fractions', *Proceedings of CERME4*.
- Hecht, S. A.: 1998, 'Toward an Information-Processing Account of Individual Differences in Fraction Skills', *Journal of Educational Psychology*. 90 (3), 545-559.
- Hunting, R. P. and Davis, G.: 1997, 'A computer- based teaching experiment as a source of potential learning environments for classroom teaching of fractions', In H. Mansfield, N. A. Pateman, & N. Bernarz (Eds.). *Young children and Mathematics: Classroom contexts and curriculum*, Adelaide: Australian Association of Mathematics Teachers, pp. 130-140
- Hunting, R. P., Oppenheimer, L. M., Pearn, C. S. and Nugent, E.: 1998, 'How six grade students explain connections between common and decimal fractions', In C. Kanes, M. Goos, & Warren, E. (Eds.). *Teaching mathematics in new times*.

Proceedings of the 21st Annual Conference of the Mathematics Education Research Group of Australasia. Rorua, N.Z. The Mathematics Education Research Group of Australasia Inc. pp. 239- 246

Laborde, C. : 2004, Instrument et processus d'instrumentation. *M2-EIAHD UE 3*. Université Joseph Fourier, Grenoble, pp. 1-8.

Lagrange, J.; Artigue, M.; Laborde, C. and Trouche, L. : 2001, A meta studio on IC Technologies in Education. In M. Heuvel-Panhuizen van den (Ed.), *Proc. 25th Conf. of the Int. Group for the Psychology of Mathematics Education*. Utrecht, The Netherlands: PME. Vol. 1, pp. 111-122

Litwiller, B. (Ed.): 2002, *Making Sense of Fractions, Ratios and Proportions, 2002 NCTM Yearbook*. NCTM: Reston, VA.

Lozano, M. D. 2004, *Characterising Algebraic Learning: an enactivist longitudinal study*. Unpublished doctoral dissertation, University of Bristol.

Lozano, M. D., Sandoval, I. T. and Trigueros, M. 2006, Investigating mathematics learning with the use of computer programmes in primary schools. In Novotná, J., Moraová, H., Krátká, M. and Stehlíková, N. (Eds.) *Proc. 30th Conf. of Int. Group for the Psychology of Mathematics Education* Prague, Czech Republic: PME., Vol. 4, pp. 89-96.

Mack, N. K.: 1998, 'Building a Foundation for Understanding the Multiplication of Fractions', *Teaching Children Mathematics*. 5 (1) 34-38.

Mariotti, M. A.: 2001, Introduction to Proof: The mediation of a Dynamic software environment. *Educational Studies in Mathematics* 44, 25–53.

Maturana, H. and Varela, F.: 1992, *The Tree of Knowledge: The Biological Roots of Human Understanding*. Revised Edition, Boston, Shambala.

Moss, J. and Case, R.: 1999, 'Developing Children's Understanding of the Rational Numbers: A New Model and an Experimental Curriculum', *Journal for Research in Mathematics Education*. 30 (2), pp. 122-47.

Rabardel, P.: 1999, Eléments pour une approche instrumentale en didactique des mathématiques. *Actes de l'école d'été de didactique des mathématiques*, Houlgate 18-21 Août, IUFM de Caen, pp. 203 – 213.

Reid, D.: 1996, Enactivism as a Methodology. In L. Puig and A. Gutierrez (Eds.), *Proc. 20th Conf. of the Int. Group for the Psychology of Mathematics Education* Valencia, Spain: PME, Vol. 4, pp. 203-209.

Taube, S.: 1997, 'Unit partitioning as a mechanism for constructing basic fraction knowledge: Testing hypothesis', Paper presented at the *Annual Meeting of the American Educational Research Association* (Chicago, IL, March 24-28). ERIC: ED 407 265.

- Tzur, R.: 1999, 'An Integrated Study of Children's Construction of Improper Fractions and the Teacher's Role in Promoting That Learning', *Journal for Research in Mathematics Education*. 30 (4), pp. 390-416.
- Varela, F.: 1999, *Ethical Know-How: Action, Wisdom and Cognition*. Stanford, Stanford University Press.

MEMORIZING ALGEBRAIC FORMULAS: THE SUPPORT OF A MICROWORLD

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This paper presents some phases of a remedial intervention in algebra in three 9th grade classes. The experiment was devoted to pupils who showed difficulties in memorization and application of the main products formulas. First, we planned an experiment based on the use of an interactive learning environment, the Aplusix microworld, whose main feature is to show students if they have made mistakes. However, since the feedback of the software seemed not to be enough, we integrated the microworld with a new artefact, the Help Window, aiming at supporting the solution process when interacting with the microworld. In this paper we will mainly analyze the role played by this artefact.

INTRODUCTION

In 9th grade mathematics classes in Italian schools, almost the whole year (the first year of upper secondary school) is devoted to algebraic calculation. A lot of time is spent in training activities: students practise the main ‘rules’ of expanding and factorizing. Nevertheless, all this work seems to bear little fruit, for teachers often complain about pupils' poor performances and their difficulties in developing even the basic competences to accomplish such tasks. Difficulties are usually categorized as memory failures in using the formulas of the main products (second, third power of a binomial, difference of squares...). Considering the memorization process led us to hypothesise possible obstacles students might encounter there. In our opinion, trying to cope with the memorization process requires taking into account at least two obstacles of different origin: on the one hand, the intrinsic difficulty of the memorization process (Norman, 1988), on the other hand, the particular *didactical contract* (Brousseau, 1997) related to algebraic calculation which does not support the memorization process but rather interferes with it.

THE MEMORIZATION PROCESS AND THE DIDACTICAL CONTRACT

It is doubtless true that most teachers consider the ability to deal with algebraic calculation, both in a correct and in a fast way, as a fundamental competence to be mastered. More precisely, the need for speeding calculations up as much as possible leads teachers to encourage the memorization of formulas instead of devoting time to understanding what manipulating an expression means. Even if questionable, this approach seems difficult to eliminate.

Once pupils are introduced to the main rules of expanding and factorizing, that is to the formulas of the main products, teachers almost immediately require both the memorization and the correct application of the formulas. As a consequence, a shift

occurs and tasks requiring algebraic manipulation (expand or factorize) become tasks concerning memorization, that is tasks aimed at checking if a certain formula was correctly memorized and can be applied straightforwardly. This means that a student who cannot remember the specific formula needed for solving an exercise remains blocked without any alternative strategies to carry it out.

The traditional intervention of the teacher to help students overcome their errors consists in reminding them of the need to memorize the formula. This is not only usually not effective, but also may add a deeper sense of frustration to students who do not get good results even after the second or the third ‘explanation’ by the teacher.

We hypothesise the need to intervene on two different levels: on the one hand memorization should not be separated from understanding (cognitive level), on the other hand, a self-consciousness of the difficulties encountered while committing formulas to memory should support the memorization process itself (metacognitive level). As far as the cognitive level is concerned, we assume that giving the students more opportunity to better understand a formula determines a change in its memorization. As regards the meta-cognitive level, we assume that only when students become personally conscious of their errors and are convinced that they have to modify their behaviour, are they effectively in a situation to overcome them (Zan, 2002). The crucial word becomes ‘metacognition’. Brown defines it as the

“knowledge about and control on one’ s own learning.” (Brown, 2002).

The author hypothesises that studies on metacognition can provide rich frameworks in which the difficult problem of memorization may be analysed as well. He summarizes his ideas by means of the following words:

“[...] Children do not use a whole variety of learning strategies because they do not know much about: (a) remembering (they know little about the strategies and tactics of overcoming memory limitations) and (b) monitoring (they do not think to orchestrate, oversee, plan, and revise their own learning activities). The terms remembering, monitoring, strategies, and metacognition widely replaced memory, denoting an important theoretical shift from passive to active metaphors of learning.” (Brown, 2002).

As a consequence, from our point of view, constructing a learning environment able both to break the didactical contract and to foster memorization is the only way to deal with the learning problem. We used the Aplusix microworld because thanks to its features the student is free to elaborate a calculation knowing that the software will control it providing a correctness feedback.

“APLUSIX can be viewed as a milieu for validation, in the sense given by Brousseau (1997), as the student can know if his/her answer is correct or not, without the intervention of the teacher.” (Chaachoua & al., 2004).

THE APLUSIX MICROWORLD AND THE ARTEFACT HELP WINDOW

Aplusix is a CAS (computer algebra system) which allows students to perform both arithmetical and algebraic calculations. The software allows the user to perform three different kinds of activities: *training* activities, *test* activities and *observation* activities. In the training activity one can carry out exercises prepared by the teacher (the teacher prepares files containing lists of exercises using the editor provided by the microworld and then users can load these). This activity is characterized by the feedback provided by the microworld on the equivalence between two consequent steps. More in detail, feedback is given by means of three different signs. Black lines point out that an expression is equivalent to the previous one; red crossed lines that you have not fulfilled the equivalence; blue crossed lines indicate that an expression is not well formed (i.e. you have opened a parenthesis but you have forgotten to close it) (Fig. 1). In case one tries to go ahead when the microworld gives a sign of not-equivalence, an error-message is displayed saying that it is not possible to go ahead because the current line is not equivalent to the previous one. In the test activity, instead, no feedback is provided: at each stage a single black line links two consequent steps. In addition, Aplusix offers another interesting tool: the *detached step*. This command opens a new independent working space, where new calculations can be carried out. We call the execution mode of a task in the training activity ‘with control mode’ and the execution mode of a task in the test activity ‘without control mode’. Finally, the observation activity allows the student, the teacher or the researcher to revise the solution given to a particular task step by step. Since it shows the solution in ‘with control mode’ whatever is the mode of execution, this activity is mainly effective for the student when who has worked in ‘without control mode’ given that the microworld points out some possible errors committed. For the teacher and the researcher the observation activity allows observation of the difficulties encountered and the error(s) committed by a pupil in great detail, without losing any steps.

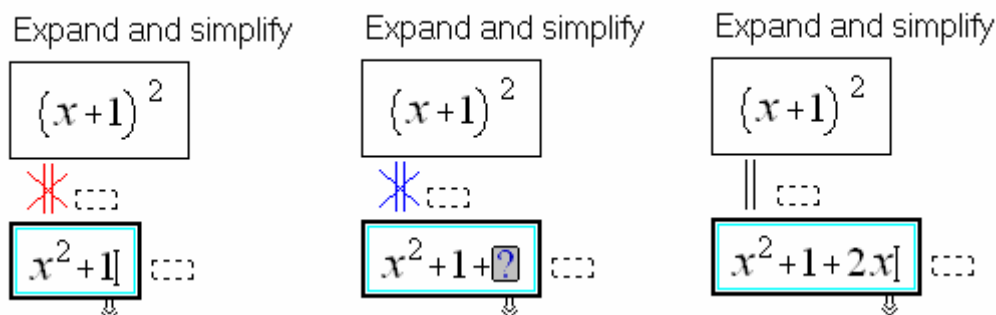


Fig. 1. Feedback provided in Aplusix environment during the exercise activity.

The Help Window is an artefact we created to be used during the interaction with Aplusix (Maffei, 2004). It consists in a list of the expanded parts of the formulas of the main products when the task is to expand an expression and in a list of the factorized part when the task is to factorize an expression (Fig. 2). Once a pupil

decides to open the window, by an icon at the bottom of the microworld worksheet, the window remains available for 60 seconds without any possibility to write on the worksheet unless one decides to close it in advance. A penalty is counted every time a student uses the Help Window. The short time given to choose a formula in the list and the penalty in consulting the window should encourage the students to memorize the formulas as soon as possible.

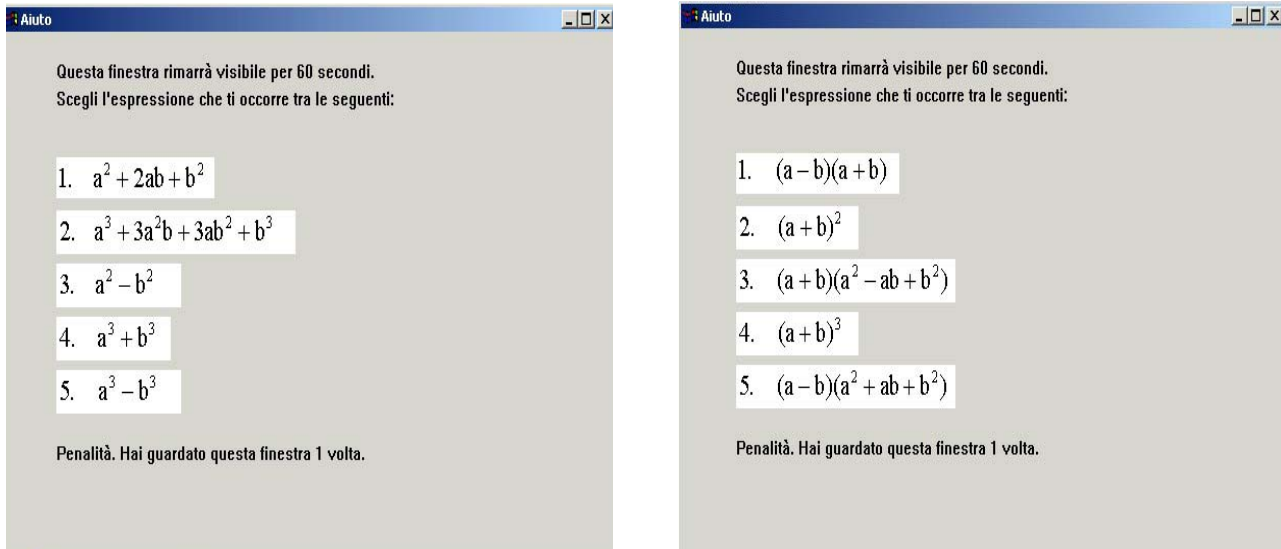


Fig. 2. To the left, the Help Window in case of expanding tasks; to the right, the Help Window in case of factorizing tasks.

THE THEORETICAL FRAMEWORK

As said above, the presence of a correctness feedback is the main feature of Aplusix and constitutes the key element on which the rationale of our remedial intervention is based. As far as feedback is concerned, the need emerges for a reference framework suitable to classify the type of feedback in order to plan an effective experiment based on the use of such an artefact. Butler and Winne (Butler & Winne, 1995) suggest a frame with the aim of explaining how feedback acts as an activator in the learning process. From their point of view, the most effective learning is *self-regulated learning* and they explore the role of feedback in acquiring self-regulation. The term self-regulated learning (*SRL*) emphasizes the need for students to acquire autonomy and responsibility in the learning processes. As a general term, it embeds research on cognitive strategies, metacognition, and motivation in one coherent construct that emphasizes the interplay among these forces. It highlights how the ‘self’ is the agent that establishes learning goals and strategies and how this engagement influences the quality of the learning process. Then, according to Balzer & al.’s findings (Balzer & al., 1989), they argue that external feedback not only gives a specific output about the task or a hint to perform it, but in doing so may enhance learners' self-consciousness and, as a consequence, learners' engagement in different tasks. Briefly, the function of external feedback consists in generating the development of internal feedback. This

external/internal exchange is not always internally oriented but flows in the opposite direction as well. For example, when external feedback is interiorized as internal feedback, then the acquired self-control may determine on the one hand a different interpretation, when confronted by the same external feedback, and on the other hand may alter knowledge and beliefs that might influence subsequent self-regulation. According to Butler and Winne, this process needs a sort of activator to develop: they called it the *act of monitoring*. Obviously, it strongly depends on the boundary conditions, that is, in our case on the effects derived from the external feedback. How well do students monitor their own processes? What difficulties can emerge in monitoring? These questions naturally arise. Since the crucial phase, the act of monitoring, is generated by external feedback, the authors go into more detail giving a classification of the kinds of feedback.

Generally speaking, an *outcome feedback* is something showing whether or not the result, or an intermediate step, is correct, while a *cognitive feedback* (Balzer & al., 1989) adds a piece of information aiming at helping the correct accomplishment of a given task. They argue that a cognitive feedback contributes to trigger self-regulated learning more than an outcome feedback.

THE RESEARCH STUDY

According to our hypotheses, the use of an *interactive learning environment* (Nicaud & al., 2006), the Aplusix microworld (Nicaud & al., 2004), was planned considering it as a tool able both to break the didactical contract and to carry out the memorization of the algebraic formulas. Moreover, as we mentioned above, in order to adapt the learning situation to the students' needs as much as possible we build an artefact, the Help Window, running at the same time as the microworld, to provide additional help to students.

The entire experimental activity followed up a twofold aim. On the one hand, there was the aim of a didactical goal consisting in the memorization of the formulas of the main products. Memorization developed through a process which favoured the retrieval of specific skills in algebraic manipulation, that is aiming at a memorization without losing, rather than, on the contrary, consolidating, the algebraic meaning (Maffei & Mariotti, 2006c). On the other, there was a research goal concerning the study of the role played by Aplusix in reaching the didactical goal: if, and if yes, how interacting with the tool may help to overcome the encountered difficulties.

In particular, attention focused on investigating the functioning of the tool, both in the cognitive processes involved in formula memorization, and in the meta-cognitive processes related to becoming aware of one's own difficulties and to managing one's own resources to improve calculation performances.

The teaching experiment

Sixteen pupils out of eighty-four (the number of pupils belonging to the three 9th grade classes involved in the experiment) needed a remedial course. All of them took part in specific activities aiming at helping them to overcome their difficulties. The teaching experiment consisted of three different phases, each of them having a specific objective (Mariotti & Maffei, 2006a, 2006b).

The first one, called “Alternative to memorization”, was aimed at fostering the use of the distributive law as an alternative strategy when the required formula is not available to memory. In fact, once the student remembered the meaning of raising to the second or to the third power, solving an expand exercise required only the correct application of the distributive property. Our hypothesis is that the repetition of the same calculation schema, again and again, should consolidate the meaning of algebraic calculation but at the same time make the need for shortcuts emerge, motivating the use of the formulas by a personal need to speed up the calculation. As a consequence, formulas may acquire their real mathematical sense, overcoming the mere sense of conforming to a teacher’s requests (Maffei & Mariotti, 2006c).

At this point, the second phase, “Memorizing formulas”, could start with the goal of supporting formula memorization. When asked to calculate by using a formula, pupils were asked to perform a twofold task: firstly to retrieve the correct formula, then to apply it in a correct way. This process was sustained (besides by the microworld) by an external support to the software: the Help Window. In this window pupils found the list of the expanded parts of the main formulas and only had to choose the specific expression they needed.

Finally, the third phase, “Recognizing formulas”, aimed at assisting pupils in reaching a good level of structure sense, that is at recognizing the structure of a main product in the exercises given. Again, a Help Window was added to the microworld. In this phase, the window contained the factorized part of the formulas. Each of the three mentioned phases finished with two tests: a test in Aplusix environment and a test with paper and pencil. During each phase we collected pupils’ commentaries by means of written reports, questionnaires and interviews. For reasons of brevity we will call the Aplusix microworld integrated with the artefact Help Window ‘Aplusix-W’.

The endpoint of the first phase

The first phase, “Alternative to memorization” was considered completed when the need for speeding up expanding tasks emerged from the students, since the sense of manipulation had been well acquired. This turning point was nicely expressed by Mattia when, in front of an expanding task, he finally felt fed up with repeating the same schema, based on the use of the distributive law, and says:

“And now, you wouldn’t believe that I’m making the multiplication again! Now, I’ll make it with a and b and then you’ll see that I learn the right one” (Mattia).

Mattia seems to feel the necessity of ‘using a formula’. He reformulates the task clearly separating the two main sub-tasks: retrieving the formula and applying it. As outlined before, the second phase of the remedial intervention was devoted to foster these two processes.

The rationale of the second phase

We exploited the rich theoretical framework, whose main points have been described in the previous section, in order to plan this phase according to the specific didactical problem related to the memorization of formulas. While the feedback provided by Aplusix can be considered as an outcome feedback since it checks the solution given (step by step and at the end of the task), the Help Window adds extra information helping to correctly carry out the task in case of error or impasse. We hypothesise that the combination of both types of feedback, the outcome and cognitive feedback, may support the twofold task. In fact, we can expect that the two different types of feedback may support the memorization process by acting at different levels. The outcome feedback acts first by showing an error in the solution sequence. Then, the output given triggers the request for consulting the Help Window: the cognitive feedback. Actually, the cognitive feedback reveals its effectiveness only when pupils can recognize, completely or at least partially, the expanded part of the formula, and so it becomes possible to match the displayed expression with its factorized part. The competence to recognize should have been gained during the first phase of the remedial intervention, “Alternative to memorization”. Once opened, the fact that the window remains visible for a short interval of time (60 seconds), in which no action is allowed apart from closing it in advance, should lead to an attempt at memorization of the recognized formula. In addition, the penalty counted each time one resorts to the assistance contributes to motivate a pupil to memorize the formulas.

RESULTS AND DISCUSSION

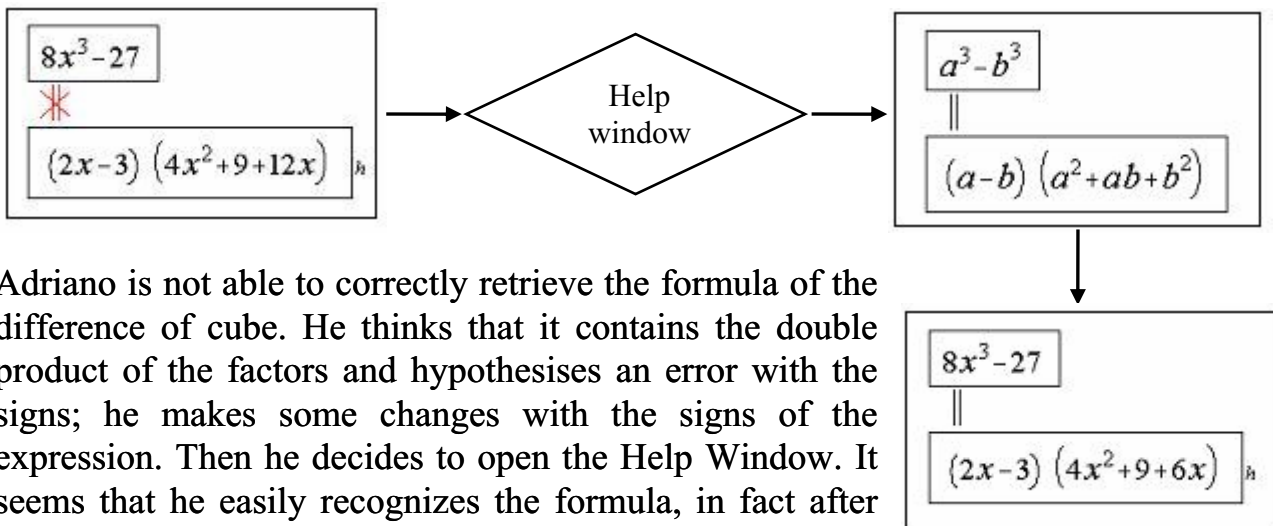
Besides the encouraging results concerning pupils’ performances in algebraic calculations (Maffei, 2004), the analysis of the data confirms the strategic role played by Aplusix-W in fostering the memorization process, as shown by the following excerpt, drawn from pupils’ written reports on their work. Renéé writes:

“The aids were useful to me because those that I couldn’t remember well and that I used to check in the independent line, were now given to me” (Renéé).

This girl clearly explains that the load her mind had to carry is now accessible by clicking on a button that opens the Help Window. We can consider the artefact as an external support fostering the internal function of memorizing.

In the following section a description of the main use’s schemes of the Help Window is given. The first example shows the basic use of the window when a pupil is facing the iteration of a negative outcome feedback.

Protocol n. 30 (Adriano) – Scenario R2_2-Phase 4 Exercise 1



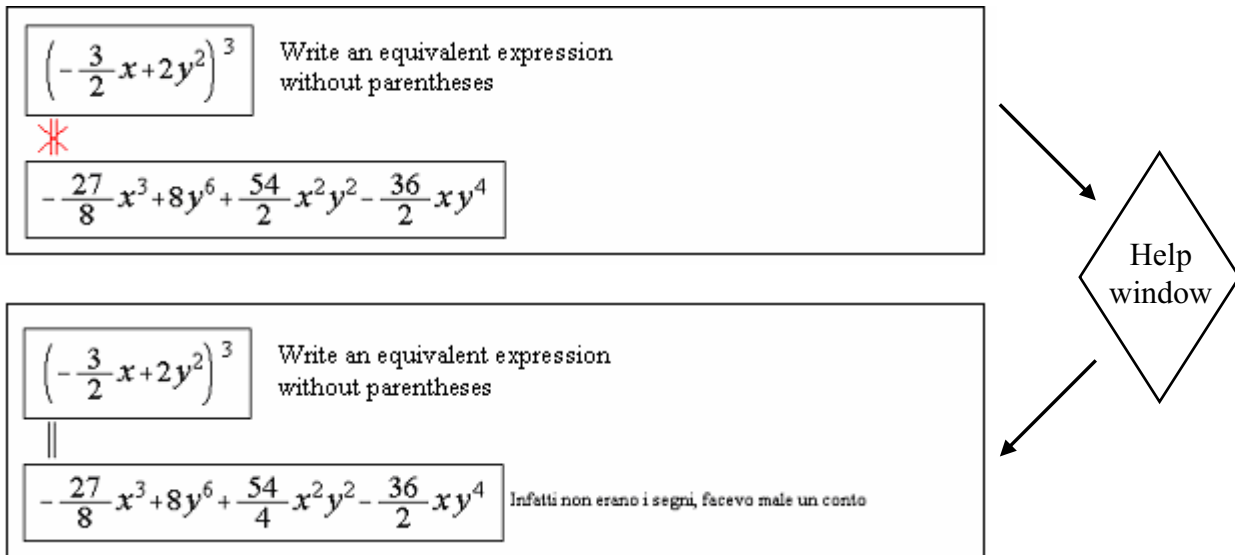
Adriano is not able to correctly retrieve the formula of the difference of cube. He thinks that it contains the double product of the factors and hypothesises an error with the signs; he makes some changes with the signs of the expression. Then he decides to open the Help Window. It seems that he easily recognizes the formula, in fact after having closed the window, he immediately writes the whole formula in the detached step (he has seen only the factorized part of the formula), then he applies it correctly. We think this strategy reveals that Adriano exploits the detached step as additional support (probably interiorized in the first phase of the intervention) to memorization. He does not seem confident in his memory yet. Other schemes of uses of the Help Window aim at preventing the red crossed lines showing the presence of errors. In these cases an acquired self-consciousness of the reached level of memorization is evident. In a questionnaire at the end of the experiment Alessio, mentioned above, in answer to the question “Did you memorize any strategy using Aplusix?”, replied:

“I have used the strategy to look often at the Help Window because in one way or another it was always useful. With the square of a binomial and the difference of squares, I have no problem; I have some hesitations, even if small, with the cube of a binomial and the product of polynomials. The help was very useful, but I think it would be very penalizing in an evaluation test” (Alessio).

Alessio seemed to make a meta-analysis of his own cognitive and metacognitive processes. In particular he was capable of self-evaluation both in terms of what he has not yet achieved (small hesitations) and contribution coming from the cognitive feedback. The last example we present supports the fact that an artefact may be used differently by the same pupil and it can be effective even if it cannot be easily detected.

Protocol n. 26 (Mattia) – Scenario R2_1-Phase 2_Exercise 3

When Aplusix showed Mattia that he had committed an error, he started to correct the expression in different ways. At each attempt he changed the signs of the monomial, but the red lines still remained. As a last chance, he opened the Help Window, looked at the list of expressions, and coming back to the solution sequence, made the proper corrections.



How can we interpret Mattia's behaviour? How did the Help Window help Mattia? We can hypothesize that keeping Mattia from working while he was looking at the list of expressions for a lapse of time led him to reflect. It seems that he overcomes the first impression based on considering the management of signs as his weak point, and acquires consciousness of not having yet memorized the formula, as he believed he had since the beginning. At the end, Mattia writes in a report:

"As regards the formulas, I knew them already but not well (sometimes I obtained them in the detached step). Now I really know the formulas, I have them stamped in my mind" (Mattia).

This comment clearly shows what we argued before about the need for a combination of the two different types of feedback in order to promote self-regulated learning. In fact, the outcome feedback acts as a spark that triggers a process in which the cognitive feedback provides the hint to correctly accomplish the task. In the meantime, having no possibility of copying the formula (when the window is still open) triggers the memorization process. The shift from the formula, displayed in the window, to its application iterated many times in the worksheet seems to positively affect memorization. In particular, the separation into two consequent phases, retrieving the formula and applying it, is underlined by the fact that the Help Window gives information only about the formula, while the students remain completely in charge of the second phase.

CONCLUSIONS

We attribute the effectiveness of the remedial intervention we have presented to the complementarity of the two processes derived from the two different kinds of feedback. In our didactical goal, consisting in the memorization of the formulas of the main products, cognitive feedback is essential in giving the message of what one has to learn. It would remain unexploited, however, without the outcome feedback, which is in charge of detecting the need for the cognitive feedback. In fact, the outcome feedback, in checking the calculation, leads pupils to overcome the errors in an

autonomous way: pupils may try to correct them without any additional help, or they can resort to the Help Window.

REFERENCES

- Brousseau G. (1997).** Theory of Didactical Situations in Mathematics. Dordrecht: Kluwer
- Brown, A.L. (2002).** Design Experiments: Theoretical and Methodological Challenges in Creating Complex Intervention in Classroom Settings. *The Journal of the Learning Science*, **2(2)**, 141-178.
- Balzer, W.K., Doherty, M.E., O' Connor, R. (1989).** Effects of cognitive feedback on performance. *Psychological Bulletin*, **106**, 410-433.
- Butler, D.L. & Winne, P.H. (1995).** Feedback and Self-Regulated Learning: A Theoretical Synthesis. *Review of Educational Research*, **65(3)**, 245-281.
- Chaachoua, H., Nicaud; J-F., Bronner, A., Bouhineau, D. (2004).** APLUSIX, a learning environment for algebra, actual use and benefits. *Proceedings of the 10th ICME Conference* [<http://www.icme-organisers.dk/tsg09/>].
- Maffei, L. (2004).** Un intervento di recupero in algebra. *Tesi di Laurea in Matematica*. 'Università degli Studi di Pisa'.
- Mariotti, M.A. & Maffei, L. (2006a).** Difficoltà in algebra: un intervento di recupero. Parte I: Il problema didattico e presentazione dell'ambiente software. *La matematica e la sua didattica*, **1**, 81-99.
- Mariotti, M.A. & Maffei, L. (2006b).** Difficoltà in algebra: un intervento di recupero. Parte II: Risultati e discussione. *La matematica e la sua didattica*, **2**, 223-245
- Maffei, L. & Mariotti, M.A. (2006c).** A remedial intervention in algebra. *Proceedings of the 30th PME Conference*, Prague, CZ.
- Nicaud, J-F. (2004).** Mixing microworld and Cas features in building computer systems that help students learn algebra. *International Journal. of Computer for Mathematical Learning*, **9(2)**, 169-211.
- Nicaud; J-F., Bouhineau, D., Chaachoua, H., Trgalova, J. (2006).** Developing Interacting Learning Environments that can be used by all the classes having access to computers. The case of Aplusix for Algebra. *Les Cahiers Leibniz*, **148** [<http://www.leibniz-imag.fr>]
- Norman, D. (1988).** The Psychology of Everyday Things. Basic Book.
- Zan, R. (2002).** Episode II: Marco and Anna - Each following their own path, from Learning from learners. *Proceedings of the 26th PME Conference*, Norwich, UK.

FLEXIBILITY AND COOPERATION: VIRTUAL LEARNING ENVIRONMENTS IN ONLINE UNDERGRADUATE MATHEMATICS

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In this paper we investigate the learning environment in the online undergraduate mathematics initiative DELTA. We find that the students work alone and that their most important learning relation is to the teacher. The students need flexibility in respect to when and where they can study, and this affects their ability to have learning relations to other students. Communication on mathematical issues is difficult using computers, and the tools offered by our LMS is insufficient. It seems hard for the students to self organise their online collaboration in mathematics.

ABOUT THE DELTA PROJECT

The DELTA project consists of eight online undergraduate mathematics courses at the Norwegian University of Science and Technology (NTNU). The subjects are the same as taught on campus, but adapted for distant learners. The syllabuses and the exams are the same as in the campus courses. Each course gives 7.5 ECTS credits, and the eight DELTA courses are:

- Basic Calculus I
- Basic Calculus II
- Linear Algebra and Geometry
- Linear Algebra with Applications
- Number Theory
- Geometry
- Probability
- Statistical Methods

The typical DELTA-student is a teacher in upper secondary school who wants to become qualified as a teacher in mathematics. Most of our students are teaching economics or biology, and some of them are mathematics teachers without a formal education in mathematics (60 ECTS credits). The students live in different parts of Norway — some students live more than 1000 km from Trondheim.

The subjects taught in our courses are very traditional undergraduate mathematics courses. We use ICT for communication, flexibility and cooperation, but the use of ICT is not a learning objective in itself. DELTA is based on the use of a learning management system (LMS). The LMS is our most important communication channel, and we use it to publish texts, streamed video lessons and exercises. The students make active use of the discussion groups offered to them inside our LMS [1]. At the Norwegian University of Science and Technology we have a studio for media productions. The media centre produces and streams our video lessons. The students emphasise the video lessons as very instructive and important for their learning.

One of the DELTA project's main issues is concerning the importance of hand writing in learning and teaching mathematics online. Our LMS is not specially designed for writing mathematics, and the students scan their handwritten hand-ins and deliver via the LMS. The teachers use a pen and a tablet with their computers, and are working directly on screen when correcting and commenting the exercises. The technological problems related to hand writing is still a major problem, precluding the students from effective collaboration online.

Norway Opening Universities (NOU) [2] is a national initiative for change and innovation in Norwegian higher education. NOU supports Norwegian institutions of higher education by funding projects for developing ICT supported flexible learning and distance education courses through an annual application process. In 2006 NOU funded the DELTA project with 500,000 NOK [3].

E-LEARNING AND MATHEMATICS

The challenges in creating an online learning environment might be different when working with mathematics than in other topics. An observation by Mark Guzdial et al. (2002) supports this hypothesis: When introducing a wiki based collaboration tool in undergraduate university teaching, specific resistance was experienced when introducing the technology in mathematics and science classes, this resistance was not seen elsewhere.

There is little research on the specific problems in using e-learning platforms for teaching mathematics, on the contrary distance education in general is relatively well explored (see Andreasen 2003, for an overview). The special situation regarding online mathematics relates, we believe, to several things: For example many of the signs that goes into building mathematical discourse is not available on a standard keyboard, and the way that mathematical communication often is supported by many registers and modalities that are used simultaneously, as writing and drawing various representations on the blackboard, while talking and gesturing (Duval 2006, Rasmussen et al. 2004). Another reason could be that mathematics as an abstract topic rely more on socially negotiated meanings than other topics, and again that this negotiation might be harder to obtain online in mathematics than in other topics. This paper can be seen as a first attempt to investigate the special situation on e-learning and mathematics, and might proceed the development of a specific framework to discuss problems and potentials with elearning and mathematics.

METHOD

The research we describe serves two parallel purposes, to evaluate and further develop the e-learning initiative DELTA and to better understand the specific challenges and potentials in using an online format to teach mathematics. The project has an obligation to perform an evaluation, and the research serves also the purpose of fulfilling this obligation. The research team consists of a teacher and leader of the

educational initiative, and a researcher not, a priori, knowledgeable about the DELTA project.

Design and research

In this project we simultaneously attempt to generate general knowledge about the learning environment in online education in mathematics and to evaluate and improve the educational initiative DELTA. In this respect we are conducting design research (DBR Collective, 2003). In our investigation we are focused on obtaining a clearer picture of the learning environment of students enrolled in DELTA and on how various factors shape this environment. Nevertheless we are not, in this investigation, focused on optimizing specific lessons or tasks, even though this would be considered a very valuable side effect.

Initial interviews and questionnaire

In order to gain insights into the learning environments of the DELTA students we applied a relatively open approach. First an open interview study was conducted, mainly in order to find relevant themes for a quantitative investigation. The interview guide was developed on the basis of intuition and experience of the involved teacher and researcher and a survey on literature. This open approach is inspired by Strauss and Cobin (1998).

Three informants were initially interviewed by phone. The three were part of a reference group. The interview guide evolved around four main question areas:

- Technological and practical barriers for communication
- What media is used and for what purpose
- Learning together with peers
- The on-campus gatherings

The interviews were transcribed and coded in an open way in order to look for themes. Themes that related to the informants learning environment, and that were of value to the informants (the things they wanted to tell). The teacher's knowledge was used to support and challenge these findings. The teacher knew some of the students from the gatherings and from contact via the LMS, e-mail and phone, and had an ongoing discussion with them about their needs. He used these 'meetings' as a sort of semi-systematic data collection. The teacher's pre-knowledge, and his need for more information to optimize the learning environment, was used in conjunction with the researcher's three interviews to find relevant questions for a quantitative investigation. We choose the following six themes as being important for the design of questionnaire:

- the need for flexibility
- the feeling of isolation
- practical barriers for working simultaneously
- the small misconceptions that is only dissolved in face to face situations

- frustration caused by a three day rule [4]
- writing mathematics using computers

Our web-based questionnaire has 32 questions, some are ‘open’, but most of the questions are multiple choice. We got answers from 32 (75 %) of the students.

LEARNING ENVIRONMENTS AND ORGANISATION

The organisation, communicative situation and mutual expectations that are connected to teaching and learning are different in different settings. As relevant to the DELTA project we consider several typical institutional organisations:

- University education in mathematics, for instance as it occurs at NTNU’s on-campus teaching.
- Secondary education, since the main target group for the education are teachers in secondary education.
- Furthermore several distance-learning formats are relevant.

The typical campus university teaching is organised with a combination of lectures and classroom teaching. The social interaction amongst peers are not explicitly supported but due to the easy access to fellow students this allows them to self organise their cooperation. On a schematic level the communication in relation to teaching and learning is centred around the teacher, who communicates to everyone, but equally important is the self organised communication amongst the students. For a deeper description of the mutual (teacher, student) expectations in undergraduate mathematics using the concept of didactical contract see Grønbaek et al. (to appear).

In upper secondary education there is typically a mix between lecture teaching and explicitly organised group work. An important aspect of upper secondary education is that group work is typically facilitated and planned by the teacher, and hence not self organised, as it typically is in undergraduate mathematics.

The classical approach to distance education is more or less to provide educational material such as one or several books, a syllabus and a possibility for evaluating. Here the main interaction is directly between the teacher and a single student (Garrison, 1985). In many approaches to online teaching the role of the teacher is furthermore to moderate and facilitate a discussion between the students. The communicative situation in this case is to focus on the participation of every student in an online learning community (Salmon, 2000).

Having these aspects and examples of learning environment in mind we consider now the situation in DELTA, looking into the student answers to the questionnaire and their comments.

DATA

Teacher as a central figure

The introductory interviews gave an impression that there was an almost inherent conflict between flexibility and isolation in DELTA, meaning that obtaining the flexibility needed requires the students to work mostly alone lacking the peer contact that is usually important in university education. In respect to contact (or lack of contact) with the peers the questionnaire shows that it is not very important for the students to study together with peers (figure 1) but slightly more important to be able to contact peers when studying (figure 3). Much more important is the contact with the teachers; 59 % agrees strongly and 41 % agrees, to this importance (figure 4).

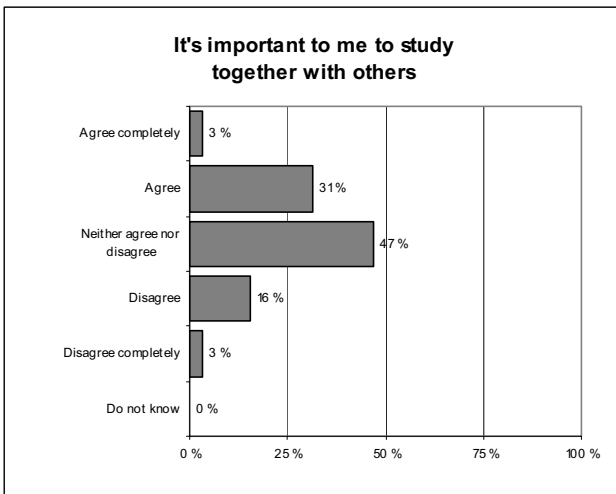


Figure 1: (Q5) It's important to me to study together with others.

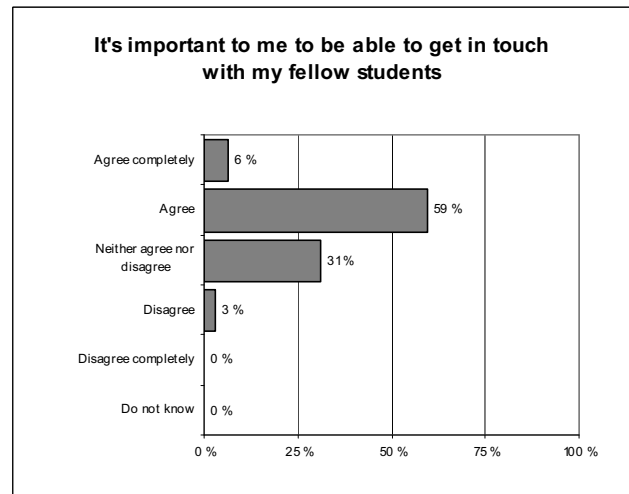


Figure 3: (Q8) It's important to me to be able to get in touch with my fellow students.

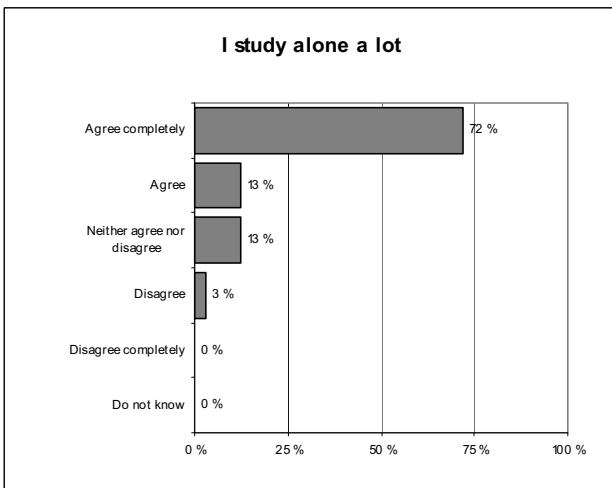


Figure 2: (Q6) I study alone a lot.

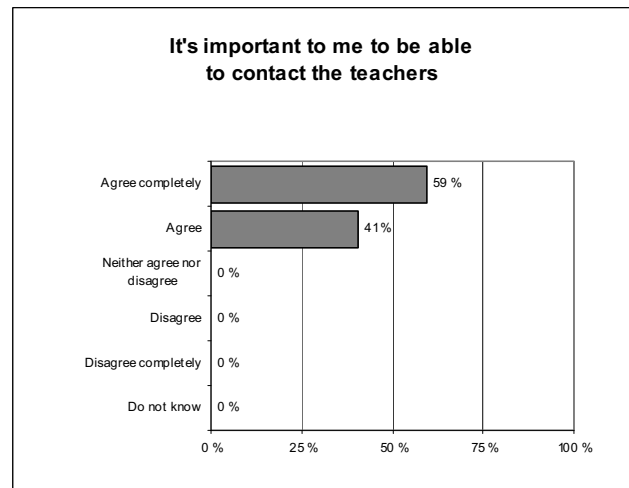


Figure 4: (Q9) It's important to me to be able to contact the teachers.

It is interesting that the difference between the need for staying in contact with peer-students and with the teacher is so large. Of course the teacher is an important figure

when studying, but these data shows that in DELTA, the teacher is the most important person for the students.

This is also reflected in the students' general evaluation of the DELTA courses where the streamed video recordings are described very positively.

It is the videos that are the backbone of the teaching. A really nice tool that I have used. In addition the discussion group for the subject is good. I have read questions/answers that have helped me.

Furthermore one thing that was revealed by the phone interviews was that the students were very keen on having prompt responses from the teacher when asking questions in the forum. The respondents felt that the teachers would hesitate to answer based on a pedagogical rationale, namely to support students' interactions, and they were unhappy with that. [4] These data all support the fact that the teacher is a very central person in the learning environment at DELTA.

Little response to the LMS discussion

Another thing that we find interesting in the responses is the low number of students that poses questions in the forum and their comments regarding motivation and barriers to pose questions.

Only a few of the students actually use the forum often for asking questions (figure 5). The questions and answers that are posted are read often (several times every week) by almost half of the students (figure 7). Furthermore many of the students weekly, or occasionally, have questions that they would like to raise but choose not to post (figure 8). Several reasons are given for this and of course lack of time and resources to pose the question are important reasons. But among the reasons are also the following (cited from students' answers to the questionnaire):

I feel I'm falling behind in relation to the expectations.

The questions are often "out-dated" because I am behind schedule ☹.

I sometimes find that my question already has been answered because someone else has posed it. In addition I feel I am lagging behind with the assignments, and concentrate on getting the assignments in rather than doing the voluntary tasks.

This means that some of the students regard the LMS space for discussion as not suitable for "out-dated" problems.

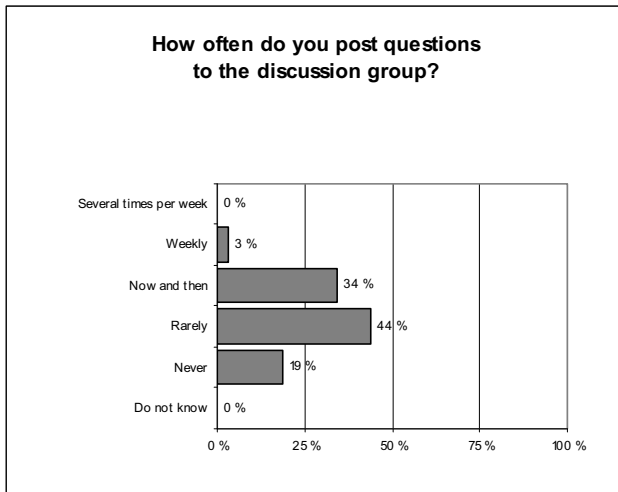


Figure 5: (Q11) How often do you post questions to the discussion group?

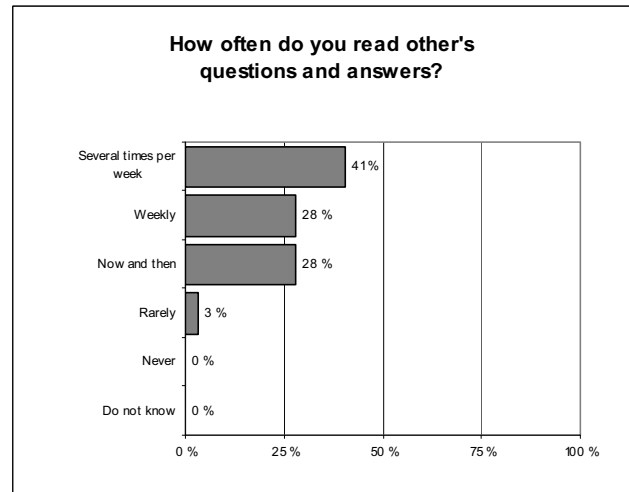


Figure 7: (Q13) How often do you read other's questions and answers?

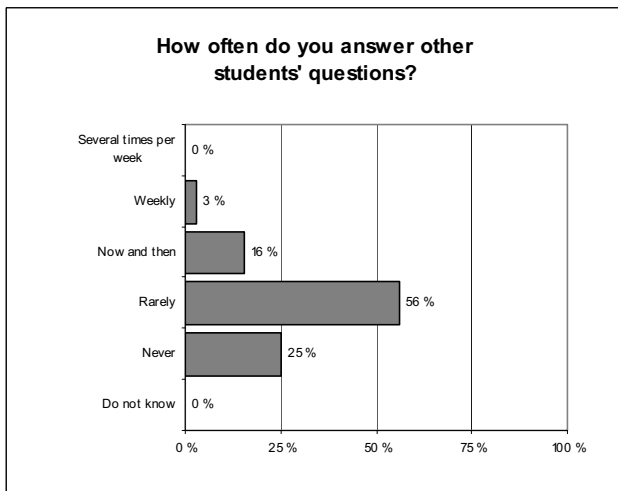


Figure 6: (Q12) How often do you answer other students' questions?

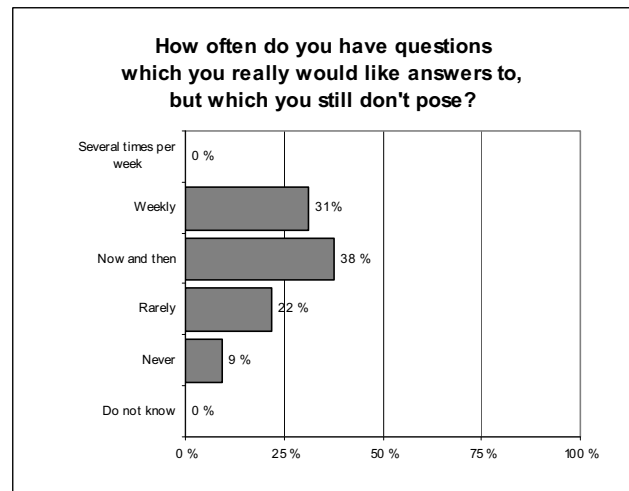


Figure 8: (Q14) How often do you have questions which you really would like answers to, but which you still don't pose?

Media and mathematics

Writing mathematical signs on a computer is clumsy (Misfeldt, 2006) and this does affect the students' ability and willingness to contribute to the online forum. For instance one student writes:

a written answer would have bee too complicated and it would not be possible with direct feedback.

In one of the questions, the students explain how they typically communicate with peer students. The most typical way is face to face or via LMS. From the students comments to the questionnaire it seems that the computer is used mainly to send and receive information and given the fact that the students choose to scan their weekly hand-ins is also important because it points to the insufficiency of for instance the e-mail format and our LMS with respect to mathematics (figures 9 and 10).

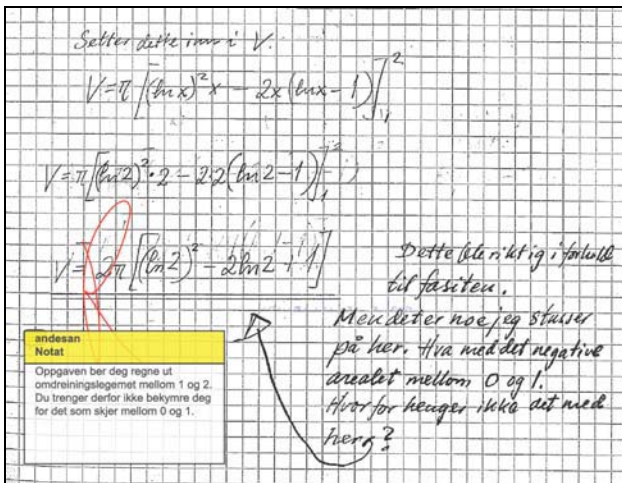


Figure 9: Excerpt from hand written assignment with student’s question to teacher and a comment from the teacher. The teacher works on screen using a pen tablet for handwriting along with the note utilities in Adobe Acrobat.

$$1 + (x - 1/(4x))^2 = 1 + x^2 - 2x(1/(4x)) + 1/(4x)^2 = x^2 + 1/2 + 1/(4x)^2 = (x + 1/(4x))^2$$

the square root of this then becomes $x + 1/(4x)$

Figure 10: Example from the calculus discussion in our LMS. This is hard both to write and to read, and with a suitable editor the student could have written this in standard mathematical notation as:

$$1 + \left(x - \frac{1}{4x}\right)^2 = 1 + x^2 - 2x\left(\frac{1}{4x}\right) + \frac{1}{(4x)^2}$$

$$= x^2 + \frac{1}{2} + \frac{1}{(4x)^2} = \left(x + \frac{1}{4x}\right)^2 \text{ and}$$

$$\sqrt{\left(x + \frac{1}{4x}\right)^2} = \left(x + \frac{1}{4x}\right)$$

ANALYSIS

The interviews left us with the impression that there is an inherent conflict between flexibility and cooperation/isolation when working with mathematics in an online environment. And the quantitative data shows that both the concerns, to obtain flexibility and to avoid isolation, are considered important by the students. The data also shows that many of the students mainly work alone (figure 2).

The students consider the contact with the teacher as more important than the contact with peer-students. In that sense the teacher is the central person having “learning relations” with each individual student. There can be several explanations for this teacher centric learning environment. The subject mathematics is different and in a sense more authoritative than for instance social sciences. This means that the online discussion format (Salmon, 2000) does not really apply to mathematics education (Guzdial et al., 2002). This does not mean, however, that the students cannot benefit from collaborating on their work, but it does mean that the discussion based format, as it is described in (Salmon, 2000) might be insufficient for supporting mathematical learning processes.

The culture around undergraduate mathematics education as it exists in campus settings can also play a detrimental role in the learning environment at DELTA. From the questionnaire we see a tendency not to pose questions in the online forum because the question is related to topics and tasks posed several weeks ago. This is actually the most typical reason that was stated in the comments to the question of why the students did not participate in the online forum. But it really does not matter that you are behind schedule, for you to pose questions. There is a big difference between

posing a question in an online forum and asking questions in an auditorium where all the attention of hundreds of students is focused on you and important teaching time is used. But the reason that the students give leaves us with the impression that they feel as if they are wasting other people's time in posing "old" questions. It is typical in campus teaching that collaboration between peer students happens automatically without the teacher organising it. The lack of day to day contact between peer students might be reason enough for the teacher to be more explicit in organising group work and collaboration, as it is also suggested in (Salmon, 2000).

The learning environment is also greatly influenced by the communicative difficulties that mathematical representations pose to online environments. It might be an extra barrier for using the online forum that it cannot handle mathematical formalism. It seems reasonable to assume that the students' difficulties in establishing well functioning collaborative relations online is partly due to their difficulties in communicating mathematically online. There is a large difference between the teachers' ability to communicate with the students through video-lectures and the students' online communicative abilities. This can be part of the explanation as to why the teacher seems very central in the learning environment at DELTA. By using the video lectures the teacher is placed in a very special position, performing well produced instructional sequences rather than participating in an equal dialogue.

CONCLUSION

This paper has investigated the learning environment in the online educational initiative DELTA. We have seen that the students engaged in DELTA mainly work alone and that their most important learning relation is to the teacher. We also see a number of reasons for this. The students in general need a lot of flexibility in respect to when and where they can study, and this affects their ability to have learning relations to other students. Mathematics is an authoritative discipline and, to some extent the teacher is governing the truth. Furthermore communication on mathematical issues is difficult using computers. And the teacher (giving video lectures) has better ways to overcome these barriers than the students.

Looking ahead three changes in the organisation of DELTA could be considered. Firstly the teachers might support the students' collaboration more explicitly, for instance by asking for cooperatively authored assignments. To trust the students to self organise their collaboration is insufficient in the online format. Partly because the students do not meet, and partly because they have trouble communicating using the computer. Secondly, it might be worthwhile to consider introducing online meeting programs as for instance Marratech [5] to support both teacher-student and student-student communication. And finally we have to find discussion software with a better editor for mathematical writing.

NOTES

1. We use the LMS *It's Learning* (<http://www.itsolutions.no>)
2. <http://nou.no>
3. Approx. 60,000 EUR
4. This turned out to originate from a misunderstanding. We do not go into a detailed analysis on the frustration caused by the three day rule in this paper.
5. <http://www.marratech.com>

REFERENCES

- Andreasen, L.B.: 2003, *Samarbejde i virtuelle læringsmiljøer: Med skriftlige netdialoger som interaktionsform*, The Danish University of Education, Copenhagen.
- DBR Collective: 2003, 'Design Based Research: An Emerging Paradigm or Educational Inquiry', *Educational Researcher* 32 (1), 5-8.
- Duval, R (2006). 'A Cognitive Analysis of Problems of Comprehension in a Learning of Mathematics', *Educational Studies in Mathematics* 61 (1-2), 103-131.
- Garrison D.R.: 1985, 'Three generations of technological innovation in distance education', *Distance Education* 6, 235-241
- Grønæk, N., Misfeldt, M. and Winsløw, C.: to appear, 'Assessment and contract-like relationships in undergraduate mathematics education', invited submission to Skovsmose, O. et al. (eds.), *University science and Mathematics Education: Challenges and possibilities*.
- Guzdial, M. et al.: 2002, 'When Collaboration Doesn't Work'. *Paper presented at the 5th International Conference for the Learning Sciences*, Seattle.
- Misfeldt, M.: 2006, *Mathematical Writing*, Doctoral Dissertation, Learning Lab Denmark, The Danish University of Education, Copenhagen.
- Rasmussen, C., Stephan, M., and Allen, K.: 2004, 'Classroom mathematical practices and gesturing', *Journal of Mathematical Behavior* 23, 301-323.
- Salmon, G.: 2004, *E-moderating: The key to teaching and learning online*, 2nd ed., RoutledgeFalmer, London.
- Strauss, A. and Corbin, J.: 1998, *Basics of Qualitative Research, Techniques and Procedures for Developing Grounded Theory*, 2nd ed., Sage Publications, Thousand Oaks, California.

ALGEBRAIC MODELLING USING A DYNAMICALLY LINKED GEOMETRY AND COMPUTER ALGEBRA SYSTEM

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Algebraic modelling is the process of describing situations in terms of variables and equations. If the situation at hand has a geometric interpretation this process can be supported by software that offers simultaneous algebraic and geometric representations. This can be achieved by dynamically linking a computer algebra system and a dynamic geometry system. A prototype Felix of such a system is presented and it is discussed how the new technical possibilities can influence the learning process.

ON THE IMPORTANCE OF ALGEBRAIC MODELLING

Algebra and geometry have a rich common history and often insight into problems of one domain could be gained by translating it to the other domain. With the advent of the computer this balance of the two fields has shifted somewhat to the side of algebra: Computers are algebraic machines, they calculate terms either numerically or symbolically. Therefore, using a computer for some kind of mathematics, may it be geometry, optimization or statistical analysis requires an algebraic (in the widest sense) formulation of the problem at hand. Often, this is only needed when a program is implemented, not when it is used. We all know that dynamic geometry systems give users the illusion of manipulating geometric objects directly. Nevertheless, there is an underlying algebraic model and at times the algebra scratches the surface, for example in the way computations of angles are handled.

Geometry can play an important role in the pupil's development of algebraic thinking. French (2002) devotes an entire chapter to suggestions exploiting this connection. Duval (1999) as cited by Hohenwater (2006) emphasises the importance of this connection by stating that "There is no true understanding in mathematics for students who do not incorporate into their cognitive architecture the various registers of semiotic representations used to do mathematics."

Several studies have shown that the use of multiple representations can support the learning process of algebraic concepts, see e.g. Stacey et al. (2004) and the references given there (especially in chapter 6). However, most of these studies focus on the functional aspect of the algebra-geometry correspondence. This is far from being unexpected as the functional concept of variation neatly fits with the dynamic interaction with a software system. The work reported here aims at extending this successful use of multiple representations to the relational aspects of algebra.

Observations of the practice of experts in applied mathematics (engineers) have shown that they need to be very flexible in modelling situations by equations and in interpreting equations in the given situation. This involves both a functional understanding (how does a value change if another is varied) as well as a relational under-

standing (what are the constraints imposed by the geometry of the object or by physical laws). The same holds true for the applications of mathematics at the secondary school level. In physics lessons students are required to combine the knowledge on a vertically falling object encoded as an equation $y = y_0 - \frac{1}{2}gt^2$ with the description of a horizontal movement $x = x_0 + v_0t$ by eliminating variables. Another subject is the calculation of currents, resistances and voltages in electric circuits which requires them to algebraically combine equations. A study in Germany (Beckmann 2000) revealed that many students lack the mathematical competencies to handle such physical applications.

In the literature algebraic modelling is almost completely seen from the functional point of view (e.g. Janvier 1996). While this functional approach certainly is of outstanding importance (both for applications as well as for understanding of concepts) it has the tendency to neglect relational aspects.

We conclude that algebraic modelling is an important ability that is worth to support by specialised learning environments. The pedagogical rationale is to enhance students' understanding of algebra and especially equations as a tool for modelling that allows one to express relations and study their meaning. The research questions derived from this are: a) How should a software system be designed that uses geometry to let students explore equations and foster their mental link between algebra and geometry? b) How do students work with such software, what strategies do they develop? c) What impact has the work with the software on the students' concept development?

ALGEBRAIC DYNAMIC GEOMETRY SYSTEMS

Computer tools such as computer algebra systems are perfect tools for experienced algebraic modellers but for beginners they lack interactivity. The user has to pose "what if..." questions in a very concise form (e.g. by deriving equations and plotting them in dependence of a parameter). On the other hand dynamic geometry systems are very explorative environments but they lack the power of algebra. We propose therefore an integration of these two kinds of software.

The kind of system we have in mind shall be called an algebraic DGS or ADGS for short. An ADGS shall offer an algebra window and a geometry window that interact and update each other mutually. These windows shall offer full CA and DG functionality.

The bidirectional connection shall be established on two levels:

On the level of the configuration (algebraically: coordinates) we have the operations of object creation and modification and of entering arbitrary coordinates.

On the level of the geometric relations (algebraically: equations) the user shall be able to impose, relax or modify them in geometric or algebraic form. Especially, the

user shall be allowed to enter arbitrary equations and inequalities that have to be respected during dragging.

An important point is that the dragging behaviour shall be governed by a simple rule: The user may forbid the movement of certain objects (fixed objects). The others move in such a way that the equations hold. This is, at least in principle, all that needs to be said to explain the systems behaviour.

A prototype of such a system has been called FeliX and it has been first realized in 2002 as an add-on for Mathematica, followed by a version for MuPAD in 2005.

Apart from FeliX there seems to be no other system that aims at this tight integration. Geogebra (Hohenwater 2006) is purely functional, GeometryExpressions (Todd 2006) offers (a subset of) geometric constraints and calculates loci algebraically but lacks the opportunity to enter and investigate algebraic relations freely. Yet other research systems aim at automated geometric theorem proving. None of these systems can claim to offer unlimited multiple representations.

THE PROTOTYPE FELIX

FeliX for the commercial computer algebra system MuPAD is invoked from a MuPAD worksheet and the MuPAD window is accessible all the time. Within the worksheet interface one has full read and write access to all objects, coordinates, equations, object creation operations and so on. For example, one may use the MuPAD programming language to automate certain constructions or to make animations. The MuPAD worksheet is the right place to do advanced algebraic manipulations starting from data constructed using the geometric tools. However, for the quick inspection of equations or the input of new equations it is more convenient not to have to switch the window. Therefore, all this information is present in the FeliX window itself as well. It is divided into a sub-window for geometry on the left and the combination of an object browser (for reading and changing coordinates, colours, fixed property, names) and an equation browser (to read, enter and modify equations and inequalities). Moreover, there is a MuPAD input line and output area that enables teletype style interactions with the MuPAD kernel without selecting the MuPAD window. This is convenient if the screen is too small to show both windows in a reasonable size at the same time.

To enter equations, the user, at least at the moment, has to use 1-d math input. Coordinate variables of objects are written for points P as $x[P]$ and $y[P]$, for circles as $x[C]$, $y[C]$, $r[C]$ and for lines as coefficients in the equation $ax+by=c$, i.e. as $a[L]$, $b[L]$, $c[L]$. Further objects are segments, vectors, and curves (which may be function graphs, parametric curves or implicit curves).

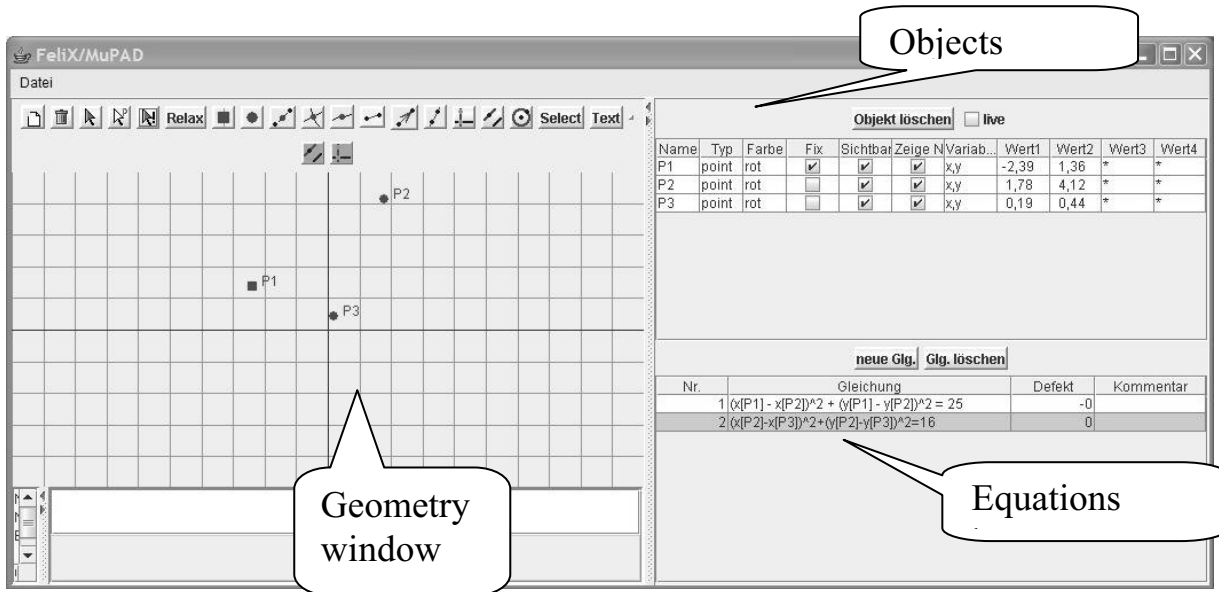


Fig. 1: The structure of the FeliX main window

In the example shown in Fig. 1 the construction consists just of three points. One may e.g. enter the equations $x[P1] + x[P2] = 2 * x[P3]$ and $y[P1] + y[P2] = 2 * y[P3]$ to declare P3 to be the midpoint of P1 and P2 (in the screenshot some other equations are shown in the equation browser). Or one changes the second equation to $y[P1] + 2 * y[P2] = 3 * y[P3]$ and observes the impact of this "rule".

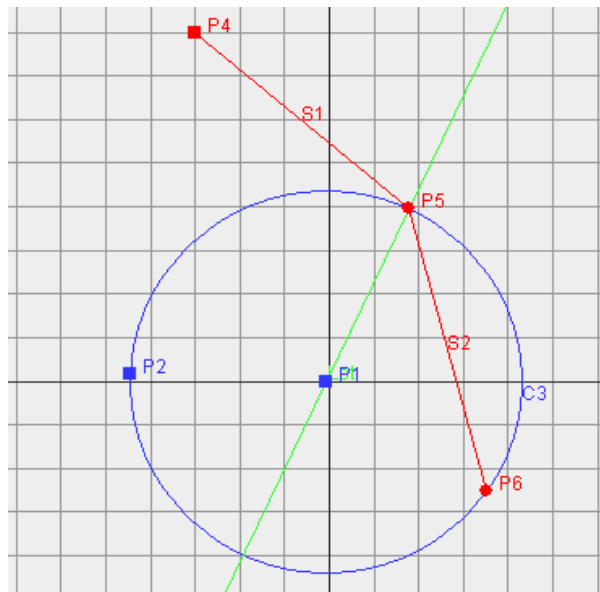
MODELLING EXAMPLES

Here we list some further examples of modelling tasks that can be handled with FeliX in a novel way. We start with Snellius' law of refraction:

Many physical situations can be modelled by a set of equations. The construction on the right shall be interpreted as a light ray S1 that hits a drop of rain and gets refracted to direction S2. Besides the obvious geometrical incidence relations we only have one physical equation that governs the physical behaviour: Snellius' law. It can be entered almost in the form found in textbooks:

$$\sin(\text{angle}(S1, \text{Normal})) = 1.4 * \sin(\text{angle}(S2, \text{Normal}))$$

Fig. 2: Rain drop



This example shows that modelling with equations in FeliX is not restricted to algebraic equations but can include transcendental equations as well.

Modelling coins

Coins on a table and circles in a dynamic geometry system behave quite differently: In the DGS the radius of a circle may change and the circles may overlap. Students that have mastered the theorem of Pythagoras can add the algebraic rules that are needed to model the physical coin situation.

If there are two circles C_1 , C_2 the following set of equations can be entered into FeliX to make the situation quite realistic:

$$\begin{array}{ll} r[C_1]=2 & \text{Fix the first radius} \\ r[C_2]=2 & \text{Fix the second radius} \\ (x[C_1]-x[C_2])^2+(y[C_1]-y[C_2])^2>4^2 & \text{Distance of midpoints of} \\ & \text{the circles must be greater than 4.} \end{array}$$

With these equations in force one may drag one circle with the mouse and push the other around the window. Moreover, one may wish to set the x-axes as a "floor" by demanding $y[C_1]>2$, $y[C_2]>2$.

This example shows that FeliX can handle inequality constraints as well. This is a very important feature, because thinking in inequalities is not adequately supported by current educational software. Related problems for students are to restrict point to move only in a half-circle or inside a lens shaped region.

A LEGO robot

It is important to model situations of interest to the students. A nice example is the analysis of the mechanism of the following little Lego "robot" which is capable of moving along a thread. Observing the real robot is very interesting because its movement is quite involved. The hand goes slowly backwards and then goes forward very quickly. Moreover, it seems not to describe a circle. We will use FeliX to analyze this mechanism.

The natural geometric model (Fig. 4) consists of a point P_1 of the axes of the motor, a



circle C_3 that describes the orbit of the end of the "elbow", a fixed point P_5 where the "lower arm" segment must pass through and a point P_7 which is the position of the robot's hand. The distance between P_4 (on C_3) and P_7 must be constraint to be of some fixed value. Plugging this information into FeliX one gets the model of the mechanism that can be moved by dragging P_7 or P_4 . The mechanism can be deformed by moving the circle or by moving P_5 . FeliX is able to calculate the equation of the orbit of the robot's hand symbolically. In the case at hand, the equation starts

with $x^6 - 4x^5 \cdot x[P1] - 2x \cdot x[P5] + 3x^4y^2 - \dots$ which illustrates that FeliX calculates the orbit symbolically including its dependence on parameters like $x[P1]$.

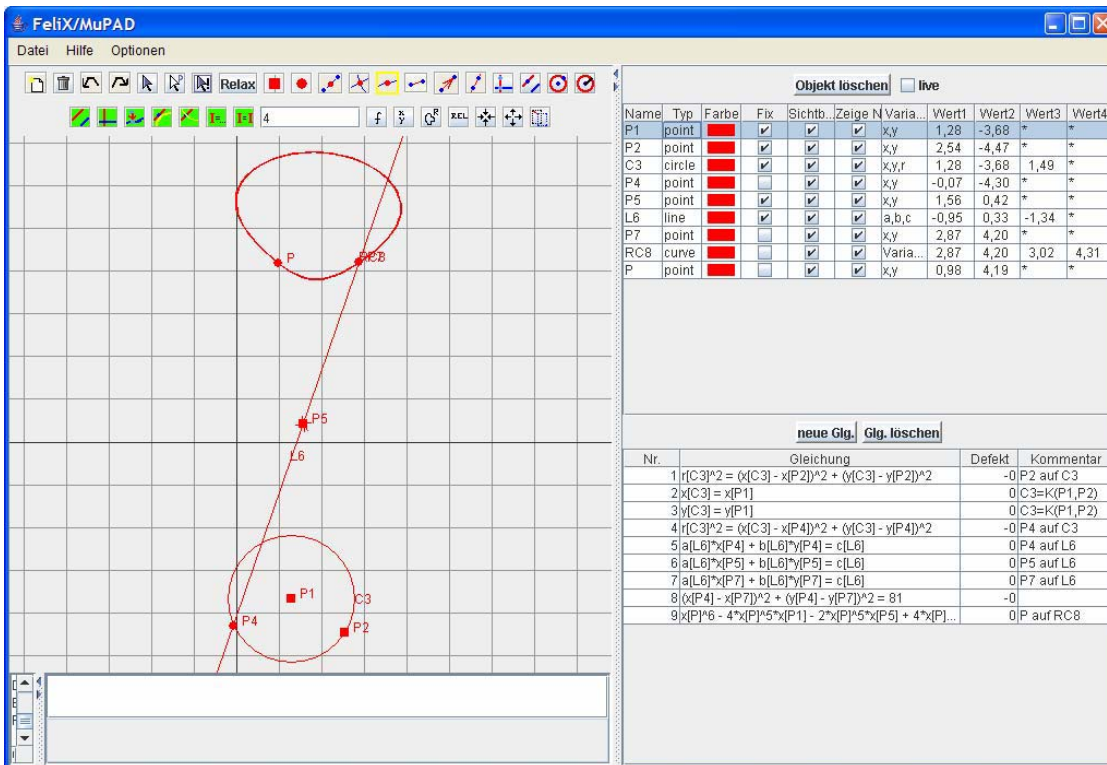


Fig. 4

A 3-bar linkage

Three-bar linkages have been studied very intensively because they generate complex mathematics from a very simple situation. Such a linkage can be realised e.g. by the following LEGO construction.

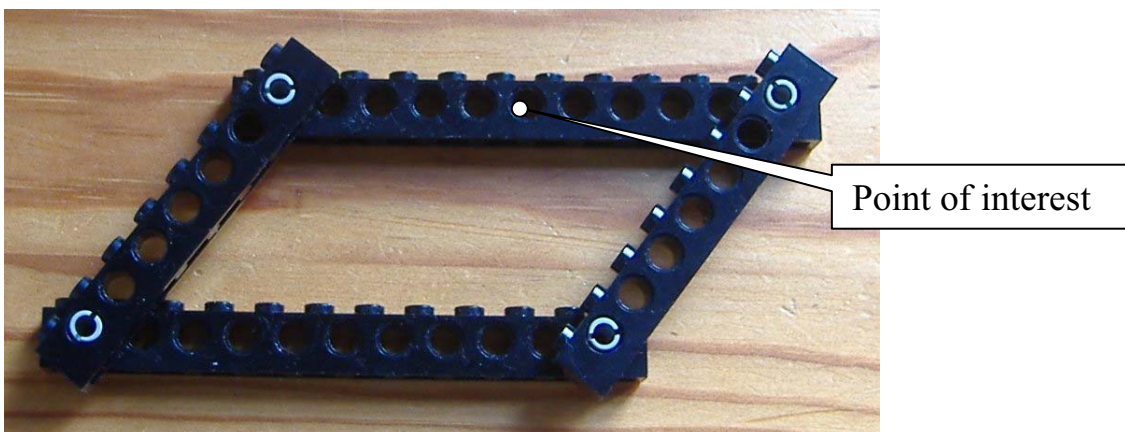


Fig. 5: A 3-bar linkage build with 4 LEGO bricks. The lower horizontal segment is considered to be fixed and is therefore not counted as a bar.

One asks for the orbit of the midpoint of the middle segment (see Fig. 5). As long as the linkage forms a parallelogram this is simply a circle, but the linkage has a second configuration in which this midpoint moves on a lemniscate. To investigate this model one simply needs to describe the distance relations realized by the rigid segments. The model then shows precisely the same behaviour as the real linkage and

FeliX can calculate the symbolic orbit. To cast it into a more readable form we substitute simple values for the coordinates of the endpoints of the lower segment:

- `subs(FelixCurves[1][3],xc["P1"]=-8,xc["P2"]=8,yc["P1"]=0,yc["P2"]=0)`

$$X^2 + Y^2 - 64 \cdot 4 \cdot X^4 + 8 \cdot X^2 \cdot Y^2 - 256 \cdot X^2 + 4 \cdot Y^4 + 768 \cdot Y^2 = 0$$

As the result is printed in factorized form it is easy to see that the orbits consist of two components, the circle and the lemniscate.

A TEACHING SEQUENCE

In the sections above the technical features of an ADGS have been illustrated. This kind of software can be used in many places in the curriculum from the introduction of Cartesian coordinates and equations to advanced algebraic geometry of curves. As a concrete example we will consider a sequence that has been taught in a 9th grade classroom after introduction of the theorem of Pythagoras. The students had modest experience with a traditional DGS (Euklid Dynageo). The introductory example was to create a point P and enter the equation $x[P2]+2*y[P]=4$. Then students had to observe how P can be moved with the mouse. They easily found that P is restricted to a straight line. They typically checked the validity of the equation at various positions of P. Only a minority used knowledge about linear equations from the 8th grade from the beginning. In subsequent tasks the students had to modify the equation such that the point comes closer to the origin or that its line has a greater slope. During these activities several students made the mistake not to modify the equation but to add a second equation. As a result, the point was rigid and didn't move at all. This observation brought up lively discussions. The validity of the equations was checked several times and finally the students explained themselves that the points is now at the intersection of the lines defined by the two equations. Then some recalled knowledge about systems of linear 2 by 2 equations and restated the explanation as the point being the unique solution of the system of equations.

This episode from the first lesson with FeliX illustrates typical learning trajectories: In a first approach the students check the dragging behaviour of objects against the equations by calculating if the equations are satisfied. (In fact, I was really surprised to see how many calculations were performed in the beginning. This shows that a very crucial phase of the instrumentation process concerns the building of confidence in the tool. By doing these vast amounts of calculations the students both build up this confidence and fostered their mental geometry-algebra link.) Only after the numerical approach students incorporate concepts like "linear equation" in their argumentation. This pattern could be observed with every new kind of problem introduced, with the "numerical phase" shortening as the confidence in FeliX grew.

The sequence went one with some nonlinear examples and with example that related two points. E.g. the students had to achieve that A is always on the left of B or that B is always exactly one unit "higher" than A. In these examples the facility of FeliX to declare some objects as fixed so that they don't move in response to the movement of

other objects was introduced. This technique was very quickly adopted by the students and most of them used it systematically to explore the situation. This observation, the episode with the rigid point subject to two linear equations and further observations e.g. with triangles with sides of fixed lengths show that in a ADGS environment the concept of “degrees of freedom” becomes crucial for the understanding of the behaviour of the construction. It is interesting that the physical term “degrees of freedom” is much better suited to describe this situation than the static mathematical term of “dimension of a solution manifold”.

In the next part of the sequence the students had to model geometric distance problems. This started of course with the basic problem of setting the distance between two points to a prescribed value. As can be expected, the students at first made no link to the theory learned before but tried to add simply the x-distance and the y-distance with several students asking about the way the absolute value function can be used in FeliX. This misconception turned out not to be viable when dragging the point. Some students even used rulers on the computer’s screen to convince themselves. But then the Pythagorean theorem was used to solve this and a vast set of related problems including e.g. restricting points to lie on or inside or outside a circle, restricting a point to lie inside the intersection of two circles, making a triangle rigid using the SSS congruence theorem, exploring parabolas and ellipses as loci of points.

CONCLUSION

In this paper we have concentrated on FeliX as a modelling tool. There are other features of such a system. From a purely geometric point of view it is interesting to investigate how a construction behaves if certain constraints are set. E.g. one may investigate which kind of constraints suffice to make a quadrilateral rigid. As one does not have to specify a construction sequence, many interesting problems (e.g. trisection of an angle) can be solved without doing constructions.

A first evaluation of FeliX based upon the sequence described above took place early in 2006 with 9th graders from a German "Gymnasium". As reported the students were very engaged in formulating equations and in interpreting them. In a final free-answer questionnaire they stated their impression that they better understood what equations between the coordinates of points express. Further results were that they judged FeliX to be easily usable and that the link between algebra and geometry is very clear.

FeliX is free but it requires the commercial computer algebra system MuPAD (www.mupad.com) to run. FeliX can be downloaded from <http://www.ph-heidelberg.de/wp/oldenbur/felix>.

REFERENCES

- A. Beckmann: Bereitet der Mathematikunterricht auf die Mathematik im Physikunterricht vor? *mathematica didactica* 23 1 (2000), 3-23.

- R. Duval: Representation, vision, and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In: Hitt (Ed.); Santos (Ed.): Proceedings of the twenty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education Bd. 1, 1999, p. 3-26
- D. French: Teaching and learning algebra. London, New York 2002.
- M. Hohenwater: Geogebra. www.geogebra.at
- M. Hohenwater: GeoGebra – didaktische Materialien und Anwendungen für den Mathematikunterricht. Salzburg 2006.
- C. Janvier: Modeling and the initiation into Algebra. In: N. Bednarz et al. (Eds.): Approaches to Algebra. Dordrecht 1996.
- R. Oldenburg: Bidirektionale Verknüpfung von Computeralgebra und dynamischer Geometrie, *Journal für Mathematikdidaktik* 26 (2005), 249-273.
- K. Stacey et al.: The Future of the Teaching and Learning of Algebra. Boston, London 2004.
- Ph. Todd: Geometry Expressions. www.geometryexpressions.com.

COGNITIVE STYLES, DYNAMIC GEOMETRY AND PERFORMANCE IN AREA TASKS

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The aim of this study is twofold, first to investigate the relationship between individual cognitive styles and performance in geometrical tasks, and second to explore the impact that a dynamic geometry software may have on students with different cognitive styles. The research was conducted with 49 sixth grade students. We hypothesized that students who tend towards a more visual cognitive style may have more abilities in area tasks and also that they may be more positively affected by a dynamic geometry environment that matches their cognitive style. The main findings of this study provide evidence that verbalizers appeared to gain more than imagers from the graphical setting of dynamic geometry due to the development of their abilities in constructing geometrical shapes.

INTRODUCTION

Since learning is a primarily cognitive activity, it is likely to be influenced by the cognitive styles of learners. Although much work has been done in this area, little attention has been afforded to the effects of cognitive styles of students on their learning of mathematical concepts. One of the main themes that interested a number of researchers in the past and today, in regard to cognitive styles, is the fact that cognitive styles may have important implications for educational theory and practice (Riding & Rayner, 1998). This becomes even more crucial nowadays that multimedia and technology based instruction is more widely used in the mathematics classroom. One of the domains that technology has been more widely used has been the use of dynamic geometry software. It will thus be of interest to investigate the impact that dynamic geometry environments may have on different cognitive style students.

The aim of this study is twofold, first to investigate whether certain cognitive styles are related to performance in area tasks and secondly to investigate whether teaching the area of triangle and parallelogram with the use of a dynamic geometry software is more beneficial for certain style of students. It is hoped that this will shed some light whether such teaching is beneficial to all students.

THEORETICAL BACKGROUND

Definition of cognitive style

Cognitive style is an individual's characteristic and consistent approach to organizing and processing information. Cognitive styles are a subset of the general subconstruct of style, which is defined as "a distinctive or characteristic manner... or method of acting or performing" (Guralnki, 1976, p.1415). Allport (1937), defined cognitive styles as the habitual way in which an individual process different information.

However, modern research actively began in a number of laboratories within a short period with the work of Witkin (1964) and Gardner (1959). Since then a number of researchers have investigated cognitive styles. Friend and Cole (1990) have expanded the definition of cognitive styles to include the way in which the individual perceives, codes, saves and recalls information, while Riding and Rayner (1998) added to Allport's definition that cognitive style is an individual's preferred and habitual approach to organizing and representing information and subsequently affects the way in which one perceives and responds to events and ideas.

Types of cognitive styles

Different researchers identified different types of cognitive styles. Witkin (1964) distinguished between field-dependent and field-independent individuals, Gregorc (1982) between concrete-sequential, abstract-sequential, abstract-random and concrete-random. Riding and Cheema (1991) reviewed over 30 methods of defining cognitive style and concluded that most could be grouped within two fundamental independent cognitive style dimensions, the Verbal-Imagery dimension and the Wholistic-Analytic dimension. Individuals' position along the Verbal-Imagery dimension reflects the manner in which they represent information while thinking, whether as words or mental pictures, while the Wholistic-Analytic dimension reflects whether they understand a situation as a whole or see things in parts.

Within the field of mathematics education the verbalisers/imagers distinction was the one that attracted most of the attention. However, it needs to be noted that this distinction was not always referred to as a cognitive style but as preferred mode of thinking, or type of students (Lean & Clements, 1981; Presmeg, 1986). Mathematics education researchers often tried to link the verbalisers/imagers distinction to mathematical performance (for example, Kruteskii, 1976, Fennema & Tartre, 1985; Presmeg, 1986). Nevertheless, the results of the relationship between visualization and mathematical performance is actually not so clear, or at least there is a need of greater clarity since researchers looked at different age groups, mathematical topics, used different methodologies or defined visualisation and visualisers differently.

Dynamic Geometry

The use of dynamic geometry as a medium of learning has been attracting much research in the field of mathematics education. The development of dynamic geometry provides learners with many opportunities to explore and discover according to their own individual needs. The basic rationale behind dynamic geometry is that information can be presented in different forms and mainly in a visual dynamic format. However, the visual form of information may come at a price because some learners may find the dynamic and visual reasoning as a complex process (Ellis & Kurniawan, 2000). Therefore, it is necessary to see how different learners perceive the features of dynamic geometry.

Individual differences and mathematical learning

In the past decade, many studies have shown evidence of individual differences and their significance in mathematics learning using appropriate software (Lee, Cheng, Rai, & Depickere, 2005). Among these differences, cognitive styles are especially related to the manner in which information is acquired and processed. It is well documented in the literature of cognitive style that field-dependence and field-independence are strong predictors of student analytical approaches to problem-solving (Nasser, & Abou-Zour, 1997). Whether a student is field-dependent or independent is an important predictor of how the student will approach mathematics problems, analyse embedded context and restructure information. Evidence from research on the effect of cognitive styles on learning suggests that cognitive style characteristics such as perception and processing of information enhance learning outcomes (Riding & Sadler-Smith, 1992). The above studies argue that optimum learning outcomes are obtained when the instructional material can be transferred readily to learners' personal modes of representation. More specifically, studies from the United Kingdom, Canada and Kuwait found that between the ages of 11 and 14 years the wholistic-verbalisers have the lowest attainment in mathematics and imagers almost double their learning performance if they are presented with the same information as text-plus-illustration compared to text, while verbalisers are not affected (Riding & Watts, 1997).

THE PRESENT STUDY

The purpose of the study

Since the use of technology in mathematics classrooms has increased dramatically during the past two decades, critical issues such as the role of students' cognitive styles need to be addressed. Connell (1998) reported that a technological environment can enhance construction of knowledge and influence learning. Computers are able to aid in visualizing abstract concepts and to create new environments that extend beyond students' physical capabilities. Dynamic software is often employed as a fertile learning environment in which students can be actively engaged in constructing and exploring mathematical ideas (Cuoco & Goldenberg, 1996). The present study is a part of a larger research project which examines the effects of cognitive styles on students' mathematical performance. Our purpose is to investigate cognitive styles along the Verbaliser-Imager (VI) and Wholistic-Analytic (WA) dimensions. However, due to space limitations, in this paper we only refer to the effects of the VI dimension on students' performance on geometry problems through the use of a dynamic geometry software. Since we know from research that a "preferred cognitive style" exists, then matching the style with the instructional format may enhance learning (Riding & Sadler-Smith, 1992). If students can access information in a format that matches their cognitive style, then the need to reorganize in accordance with their preferred style prior to learning is not necessary. The elimination of this step in information processing presumably reduces the cognitive

load imposed by the task and enhances performance (Halford, 1993). Thus, the first hypothesis of the study is that imagers will outperform verbalisers in area tasks. The second hypothesis of the study is that imagers may benefit more from the teaching of area with the use of a dynamic geometry software, since their cognitive style matches the format of representations in this media.

Participants and procedure

Forty nine sixth graders (27 boys and 22 girls) from two intact classes of an urban school in Cyprus participated in the study. The research consisted of a cognitive style test, a pre-test, the instruction with dynamic geometry, and the post-test. The computerised VICS test (Peterson, 2005) was administered to the students during a 30 minute school period. This test was used as an independent variable to split the sample into verbal (textual) and imager (graphics) groups. After a week, students were given the area test (pre-test) which it was again administered to the students the day after the instructional program with the dynamic geometry was completed (post-test). The pre-test was used to indicate participants' prior knowledge of area and to facilitate the measurement of cognitive performance on the post-test.

The instructional program focused on the teaching of area of triangles and parallelograms with the use of the dynamic geometry software Euclidraw Jr. More specifically the aims of the instructional program are presented in the Figure 1.

<p><i>Area of triangle</i></p> <ul style="list-style-type: none"> To recognise and draw the altitudes of triangles. To understand the concept of shapes with equal area. To discover the mathematical formula for the area of triangle. To calculate the area of various triangles by applying the formula. To construct a number of different triangles with specific area. To explain the way in which the area of triangle changes when its altitude and/or base change. <p><i>Area of parallelogram</i></p> <ul style="list-style-type: none"> To recognise and draw the altitude of a parallelogram. To discover the mathematical formula for the area of parallelograms. To measure the length of altitude and base in order to calculate the area of a parallelogram on squared paper. To calculate the area of various parallelograms with the use of the formula. To construct various parallelograms with specific area. To explain the way in which the area of parallelogram changes when the altitude and/or base change.
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Figure 1: Aims of the intervention course

The topic of area was chosen, on the one hand, because it is one of the most commonly used domains of measurement in everyday life, and it is the basis for

many models used by teachers and textbooks to explain multiplication of whole numbers and fractions. On the other hand, there is evidence that both elementary and secondary school students have inadequate understanding of area and area measurement (Outhred & Mitchelmore, 2000; Kordaki & Balomenou, 2006). Thus, the teaching of area and area measurement is an appropriate topic for the use of dynamic geometry since it requires the integration of spatial and numerical concepts.

Euclidraw Jr was selected amongst other dynamic geometry software since it appears to be more appropriate and user-friendly to primary school students. Some of the Euclidraw Jr tools that have been applied during this instructional program were: (a) the dragging mode (b) direct construction of geometrical shapes (c) various ways of construction and measurement of segments and shapes (d) appearance of grid lines (e) cutting shapes tool (f) joining shapes tool. The duration of the intervention program was two weeks (eight 45-minute periods). The students had some experience with the software from previous lessons.

Tasks of the study

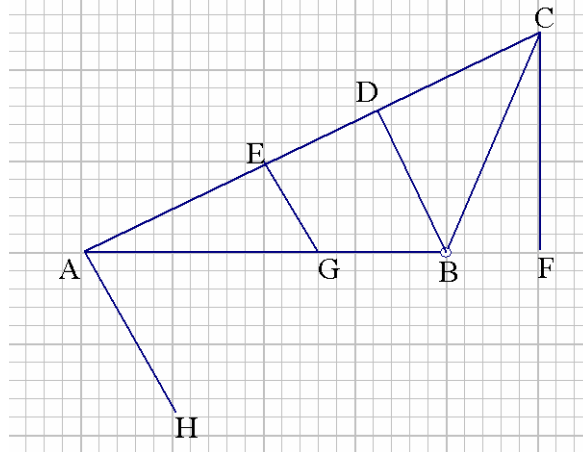
All students were assessed for their preferred cognitive style using the Verbal – Imagery Cognitive Style test (VICS test) (Peterson, 2005). The test works on the basis of response times to a battery of statements which are categorized into subsets, and a ratio for each subset is calculated. The VICS test measures the Verbal - Imagery dimension by asking which of the two shapes is bigger in real life or whether two items are man-made, natural or mixed. Both these questions are presented either with the use of icons or words. A detailed discussion of the rationale for the design of the two tests can be found in Peterson, Austin & Deary (2005). For the purposes of this study on the VI continuum, students that tended towards the verbal cognitive style were those with scores equal to or less than 1 and students that tended to be imagers were those who scored more than 1. Recently a number of researchers do not use this cut off score to make the distinction between the two groups (see for example Peterson et al., 2003a; 2003b).

The area test

The area test that was used for the purpose of this study included 24 tasks, 12 on the area of triangle and 12 on the area of parallelogram. In six of these tasks students were asked to identify triangles or parallelograms with the same area, or specific elements of these shapes such as their altitude (Recognition Tasks). There were 8 construction tasks which called upon students' ability to construct triangles or parallelograms with specific area, or draw the altitudes of these two shapes (Construction Tasks). Finally, there were 10 tasks which required students to compute the area or the height or the base of triangles and parallelograms (Computation Tasks).

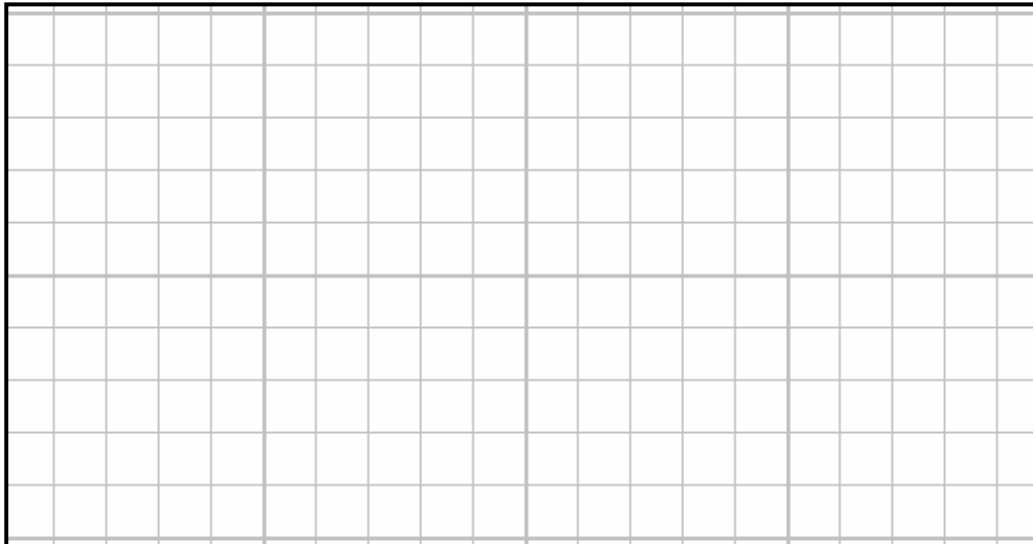
A. Recognition Task

Draw with red the segments that show the altitudes of the triangle ABC.



B. Construction Task

Construct three different triangles with the same area.



C. Measurement Task

Complete the following table for the area of triangles.

<i>Base</i>	<i>Height</i>	<i>Area</i>
<i>5 cm</i>	<i>4 cm</i>	<i>.....</i>
<i>.....</i>	<i>6 cm</i>	<i>24 cm²</i>
<i>6 cm</i>	<i>.....</i>	<i>48 cm²</i>
<i>.....</i>	<i>.....</i>	<i>50 cm²</i>

Figure 2: Area Tasks

DATA ANALYSIS

To examine the hypotheses of the study, descriptive statistics were used and multivariate analysis of variance was applied. The total score in pre and post-tests as well as in the sub-scales of the tasks (recognition, construction and computational) was used as dependent variables, while the cognitive styles of students on the VI continuum served as independent variable. In addition, the progress of students in the area test was calculated using the difference in the mean scores of the pre-test and post-test.

RESULTS

The results of the study are presented according to the hypotheses of the study. We first focused on comparing student performance in area tasks based on their cognitive styles. Through the administration of the VICS test, we split the sample in two groups, the verbalizers and the imagers. We found that on the VI continuum 29 students tended towards an imagery preference and 20 towards a verbal preference. At the same time students' scores on area tasks in the pre-test for imagers and verbalisers were compared.

Table 1 shows the means of students' performance in the area tasks (pre and post tests) and in the subscales of the test. Table 1 also shows the gain scores in the total test and the subscales as resulted from the calculation of the difference between the initial performance of students and their performance after the intervention with the use of the dynamic geometry. Finally, the last two columns of Table 1 show the results of the multivariate analysis. As can be deduced from the table, there was no significant difference between imagers and verbalisers on the pre-test total geometry scores ($F_{(1,48)} = 4.13$, $p = 0.05$), indicating that the cognitive style of students does not affect in a significant way students' performance in geometry. Furthermore, comparing scores for imagers and verbalizers on the sub-tasks of the pre-test (recognition, construction, and computation tasks) also revealed no significant difference. However, the mean score of the verbalisers on the total test and on all sub-tasks was consistently better than that of the imagers (see the means on Table 1). Hence, it may be plausible to suggest that there appears to be a pattern (although not significant) in favour of the verbalisers (Table 1). These results do not seem to confirm the first hypothesis of the study that the imagers would have a better performance in area tasks than students with other cognitive styles and contradict the findings of much of the research conducted by Riding and Watts (1997).

	Verbalisers	Imagers	F	p
	Means	Means		
Pre-test total	7.57	6.97	0.31	.58
Post-test total	12.53	9.87	4.13	.05
Benefits in total score	4.97	2.9	9.97	.00*
Recognition pre-test	0.28	0.25	0.54	.47
Construction pre-test	0.32	0.31	0.01	.92
Measurement pre-test	0.38	0.36	0.19	.66
Recognition post-test	0.49	0.41	1.61	.21
Construction post-test	0.49	0.37	2.88	.10
Measurement post-test	0.62	0.50	2.34	.13
Benefits in recognition	0.21	0.17	0.50	.48
Benefits in construction	0.18	0.06	4.51	.04*
Benefits in Measurement	0.24	0.15	3.04	.09

* Indicates statistical significance at $\alpha=0.05$

Table 1: Means of students' performance and multivariate analysis of variance with cognitive styles as independent variable

Table 1 also shows that the cognitive styles under investigation do not seem to affect students' performance in the overall area test, neither in the subcategories of the test, even after the intervention with the use of the dynamic geometry software (see post-test results). This result was somehow expected since the duration of the intervention course was a short one. Of most importance is the progress of students' performance after the intervention. Thus, a multivariate analysis was conducted with dependent variables the progress of the students from the pre-test to the post-test.

This analysis showed that verbalizers improved significantly their performance in the total score ($F=9.97$, $p=0.00$). Specifically, verbalizers gained significantly more than imagers, indicating that the progress of the verbalisers was higher than the progress of the imagers in the total score. This improvement was mainly due to the rise in verbalisers' performance in the test's construction tasks ($F=4.51$, $p=0.04$). This finding is in contrast to the second hypothesis of the study as well as to previous research indicating that optimum learning is achieved when individuals are taught in a way that matches their cognitive styles. Finally, no differences were found in computational and recognition tasks between verbalizers and imagers.

DISCUSSION

The findings of previous studies provided evidence that computer learning programs may not be suitable for all learners as a learning tool, and teachers should be aware of individual differences such as cognitive styles possessed. Teachers also should not assume that every student would equally benefit from the use of computers in educational settings (Connell, 1998). In this study, we investigated the effect of cognitive styles on area tasks, based on the notion that verbal (text-based) instruction

is best suited to a verbal cognitive style while pictures (graphical representation) suit an imagery-based cognitive style best (Riding & Rayner, 1998).

First, the findings of the present study do not suggest that cognitive style is a decisive factor in explaining the differences in area tasks between students with different cognitive styles. Second, we focused on the hypothesis that the use of dynamic geometry, which mostly provides experiences in pictorial forms, may enhance the performance of imagers in area tasks. However, we found that the effect between cognitive style and dynamic geometry is at variance with the above hypothesis.

Specifically, the results of the study provide evidence that verbalizers appeared to gain more than imagers from the graphical setting of dynamic geometry. This result may suggest that verbalizers perform best when given an instructional format enhanced with graphical features or that they are more able in dealing with area formulas. In addition, it seems that the improvement of verbalizers was mainly due to the development of their abilities in constructing geometrical shapes. This prompts research into the nature of geometrical abilities required for the improvement of students' performance in specific subcategories of problems referred to as recognition, construction and computation of geometrical concepts. Further investigations of what makes a subcategory more or less suited to certain cognitive styles are currently being considered.

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REFERENCES

- Allport, G.W. (1937). *Personality: A psychological interpretation*. New York: Holt.
- Connell, M. (1998). Technology in constructivist mathematics classrooms. *Journal of Computers in Mathematics and Science Teaching*, 17(4), 311-338.
- Cuoco, A., & Goldenberg, E.P. (1996). A role for technology in mathematics education. *Journal of Education*, 178(2), 15-32
- Ellis, R.D., & Kurniawan, S.H. (2000). Increasing the usability of online information for older users: A case study in participatory design. *International Journal of Human-Computer Interaction*, 2(12), 263-276.
- Fennema, E., & Tartre, L.A. (1985). The use of spatial visualization in mathematics by girls and boys. *Journal for Research in Mathematics Education*, 16(3), 184-206.
- Friend, C.L., & Cole, C.L. (1990). Learner control in CBI. *Journal of Educational Technology*, (November), 47-49.
- Gardner, R.W. (1959). Cognitive control principles and perceptual behaviour. *Bulletin of the Menninger Clinic*, 23, 241-248.
- Gregorc, A.F. (1982). *Gregorc style delineator*. Maynar, MA: Gabriel Systems.

- Guralnki, D.B. (1976). *Webster's new world dictionary (2nd college ed.): Classic edition*. Akron, OH: William Collins.
- Halford, G.S. (1993). *Children's understanding: the development of mental models*. Hillsdale, NJ: Erlbaum
- Kordaki, M., & Balomenou, A. (2006). Challenging students to view the concept of area in triangles in a broad context: Exploiting the features of CABRI-II. *International journal of computers for mathematical learning*, 11, 99-135.
- Kruteskii, V.A. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago. University of Chicago Press.
- Lean, G., & Clements, M.A. (1981). Spatial ability, visual imagery and mathematics performance. *Educational Studies in Mathematics*, 12, 267-299.
- Lee, C.M., Cheng, Y.W., Rai, S., & Depickere, A. (2005). What affect student cognitive style in the development of hypermedia learning system? *Computers & Education*, 45, 1-19.
- Nasser, R., & Abou-Zour, S. (1997). Effect of cognitive style on algebra problem solving using computer aided approach versus an analytical method. *International Journal of Computer Algebra in Mathematics Education*, 4(4), 359-370.
- Outhred, L., & Mitchelmore, M. (2000). Young children's intuitive understanding of rectangular area measurement. *Journal for Research in Mathematics Education*, 31(2), 144-167.
- Peterson, E.R., Deary, I.J., & Austin, E.J. (2003a). On the Assessment of Cognitive Style: Four Red Herrings, *Personality and Individual Differences*, 34, 899-904.
- Peterson, E.R., Deary, I.J., & Austin, E.J. (2003b). The Reliability of the Cognitive Styles Analysis Test, *Personality and Individual Differences*, 34, 881-891.
- Peterson, E.R. (2005). *Verbal Imagery Cognitive Styles & Extended Cognitive Style Analysis-Wholistic Analytic Test- Administration Guide*. New Zealand: Peterson, R. E.
- Peterson, E.R., Austin, E.J., & Deary, I.J. (2005). A new measure of Verbal-Imagery Cognitive Style. *Personality and Individual Differences*, 38, 1269-1281.
- Presmeg, N.C. (1986) Visualization and mathematical giftedness. *Educational Studies in mathematics*, 17, 297-311.
- Riding, R.J., & Cheema, I. (1991). Cognitive styles: an overview and integration. *Educational Psychology*, 11, 193-215.
- Riding, R.J., & Rayner, S. (1998). *Cognitive styles and learning strategies: understanding style differences in learning and behaviour*. London: D. Fulton.
- Riding, R.J., & Watts, M. (1997). The effect of cognitive style on the preferred format of instructional material. *Educational Psychology*, 17, 179-183.
- Riding, R., & Sadler-Smith, E.S. (1992). Type of instructional material, cognitive style, and learning performance. *Educational Studies*, 18(3), 323-340.
- Witkin, H.A. (1964). Origins of cognitive style. In C. Sheerer (Ed.), *Cognition: Theory, research, promise*. New York: Wiley.

MEANINGS FOR FRACTION AS NUMBER-MEASURE BY EXPLORING THE NUMBER LINE

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Construction of meanings for fraction as number-measure is studied during the implementation of exploratory tasks concerning comparison and ordering of fractions as well as operations with fractions. 12-year-old students were working collaboratively in groups of two with software that combines graphical and symbolic notation of fractions represented as points on the number line. Fractions as points and segments, ordering fractions as part of kinesthetic activities and abstracting the scaling of the numerical unit on the number line are some of the meanings developed.

THEORETICAL BACKGROUND

In this paper we report research [1] aiming to explore the mathematical meanings constructed collaboratively by 12 year-old students concerning the notion of fraction as a number-measure depicted on the number line within a specially designed computational environment. The students worked in collaborative groups of two using the ‘Fractions Microworld’ (FM) [2], a piece of software which combines graphical and symbolic notation of fractions represented as points on the number line. The research perspective and task design adopt a constructionist approach to learning (Harel & Papert, 1991), focusing particularly on students’ interaction with joint representations (Kynigos, 2002) to construct mathematical meaning.

In the research literature understanding fractions involves the coordination of many different but interconnected ideas and interpretations such as part-whole, measure, quotient, operator and ratio (Lamon, 1999). The interpretation of a fraction within part-whole relations is the first and probably the most dominant facet of the concept presented to students at the primary level, after which the algorithms for symbolic operations are introduced. Whereas algorithmic competencies in the domain of fractions are usually fairly developed, they are mainly associated with a mechanistic use of fractions in calculations while understanding is usually weaker as well as the competencies to solve problems including fractions especially in number-measure situations (Aksu, 1997). These situations of fractions are usually accompanied by the pictorial representation of a number line and students are expected to measure distances from one point to another by partitioning certain distances from zero in terms of some unit. The value of the resulting fraction in this case comprises *the number* (i.e. the rational number that the fraction represents), while the distance on the number line *the measure*. This dual reference of partitioning to quotients as well as to distances from zero can be seen as an example of the complexity of situations in which the number-measure interpretation of fractional numbers occurs.

The existing research results confirm that the measure interpretation of fractions on the number line is one of the most difficult for the students to acquire since it is related to the well documented hidden discontinuities between natural and fractional numbers (Stafylidou & Vosniadou, 2004) such as the uniqueness in the symbolic representation of natural numbers, which does not hold for fractions (i.e. several fractions can represent the same fractional number), or the density of fractional numbers depicted on the density of number line (i.e. between any two fractions there is an infinite number of fractions). As children's experience with numbers is usually based on the discrete integers used for counting, their theory of number may become increasingly resistant with age to accepting fractions as numbers that represent the continuous nature of measurable points in space. This situation is aggravated by the apparent difficulty of students to relate the number line with their real-world knowledge or some kind of external realistic grounding (Marshall, 1993). Meaningful examples that incorporate pupils' previous experiences may be difficult to derive especially at the younger ages. In the absence of a "mental model" for filling the gaps between the integers, children are likely to overgeneralize their use of counting numbers in a way which provides a formidable obstacle to accommodating their theory of number to accept fractions (Siegal & Smith, 1997). Another constraint concerns the students' confusion over the nature of unit on the number line. As fraction is usually perceived by pupils through the part-whole metaphor, the unit is considered as bigger than all fractions and many times the integer number line is treated as a unit rather than the segment from zero to one (Baturu & Cooper, 1999).

Although there are certain difficulties in conceptualizing the number line representation of fractions there is a recent resurgence of interest in the representational potential of number lines for the learning of fractions (Ni, 2001, Hannula, 2003, Charalambous & Pitta-Pantazi, 2005). Entailing "a dynamic movement among an infinite number of stopping-off places" (Lamon, 1999, p. 120), working with fractions on the number line has been considered as critical for pupils to conceptualise the different subconstructs necessary for the deep understanding of the concept of fraction in general (Hannula, 2003). Recently, the representational infrastructure of computer-based environments provided us with tools to reconsider the role of the number line in making the number-measure facet of fractions more accessible and meaningful to children. In this study experimentation with fractions is considered as more flexible and dynamic for the students due to the available tools that combine graphical symbolic notation allowing manipulation of the provided representations. Under this perspective we emphasised not on closed 'didactical goals' but on pupil's active construction of meanings as they operationalised the use of the available tools while making judgments, taking decisions and developing situated abstractions (Noss & Hoyles, 1996) during the problem solving process. The concept of fraction as measure was not considered on its own but in terms of the concepts tightly related to it, the situations in which it may be used and the available representations (i.e. within a conceptual field, Vergnaud, 1991).

TECHNOLOGY AND TASKS

In our research perspective we attribute emphasis on two aspects of the pedagogical setting that are likely to foster mathematical learning: the computer environment and the task design. The construction of a fraction in the FM is realised as a quotient of a division: the divider and divisor are selected from one horizontal and one slanted line respectively (see Figure 1). After the selection of two numbers from the slanted lines it is also provided instantly a geometrical representation of a fraction based on the Thales Theorem. Since this theorem is introduced at the secondary level, we chose to bypass this interpretation in the present study and to use it as a ‘black-box’ for the students. The horizontal line is called ‘number line’ while the slanted one is called ‘multiplication (or partition) number line’.

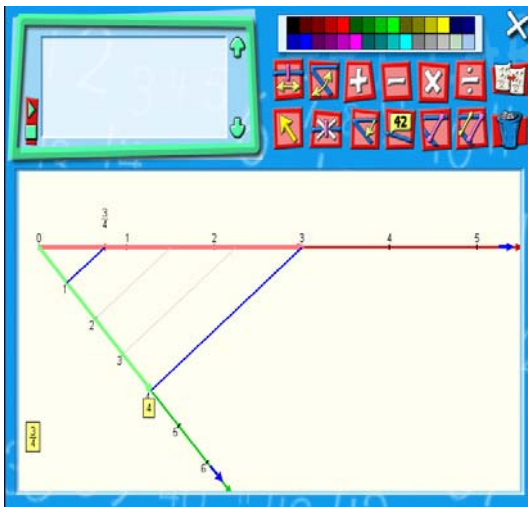


Figure 1: $\frac{3}{4}$ represented on the FM horizontal number line.

The symbolic notation of each fraction is automatically given near its representing point on the horizontal line in a post-it form. In the FM there is thus a representation of fraction as a number (resulting by the division between the numerator and the denominator) as well as a measure (represented by a point on the number-line). Other features of the microworld, like the ability to modify dynamically the size of the numerical unit (i.e. the distance between two successive integers) in both half lines and to perform basic arithmetic operations on fractions (i.e. addition, subtraction and multiplication) are parts of the visual imagery and manipulative aspects of the tool. The novel

character of the above representations can thus lead to the identification of a ‘distance’ between the mathematical objects constituting the representation of fractions in the microworld (tool design) and those found in the traditional curriculum, based primarily on the part-whole scheme. For example, the symbolic representation of fractions with a numerator equal to 1 coincides with the part-whole representation of these fractions (e.g. the position of the fraction $\frac{1}{3}$ indicates also the respective part-whole relationship, 1 part of the 3 in which the unit 1 is divided) which does not happen with any other type of fractions as represented in the FM. Moreover, the lack of any observable partitions of the numeric unit on the horizontal line can be considered as a characteristic that enhances the abstract nature of the representation of fractions as points among integers.

In task design it was adopted a perspective in which the ‘distance’ between tool design and aspects of didactic knowledge of fractions was considered as a challenge to design exploratory activities which may provoke multiple pupil’s responses concerning comparison and ordering of fractions as well as operations with fractions.

This decision was also reinforced by the fact that the FM only represents fractions in specific ways and does not signal some kinds of mistakes by means of visual feedback. This characteristic of the tool gives space for pupil's interpretations of the given feedback which can be seen as a point of negotiation among the students rather than as a closed answer. The above choices led to the design of two strands of problems (see Tables 1 and 2) based on integrating the representation of fractions on the number line with places in an everyday context. In pupils activity thus the mathematical nature of fractions was planned to integrate with their use to measure quantities in authentic problem situations in which the move in different points on the number line was connected with the idea of persons covering certain distances.

- (1) George's house is 1 kilometer far from his school. On his way to school he sees a square at $\frac{1}{2}$ km, his friend's Chroni's house at $\frac{1}{3}$ km and a sweetshop at $\frac{1}{6}$ km. Can you say in which order he sees them when coming back home?
- (2) (a) Constantina's school is 1 km away from her house. On her way to school she sees a kiosk at $\frac{6}{7}$ km, a super market at $\frac{2}{5}$ km and a playground at $\frac{3}{4}$ km. Which is the order she sees them on her way to school?
- (b) Lazaros is Constantina's best friend; his house is between the playground and the super market. Can you find some fractions indicating the position of his house?

Table 1: The first strand of activities.

- (3) Efi and Constantina are friends. They meet each other at the playground. Efi says to Constantina: "You are very lucky. Your house is closer to the playground than mine." Discuss about the position of Efi's house.
- (4) Constantina says to Efi: "I think that you are lucky too. You walk only $\frac{2}{3}$ kilometers to go to school." Why is Efi lucky? Can you find the exact position of Efi's house? What is the distance of the two friends' houses?
- (5) Maria, a friend of Constantina and Efi, says: "I believe that the luckier of you is the one who walks less in going both to the school and the playground every afternoon." Who do you believe is luckier?

Table 2: The second strand of activities.

METHOD

The experiment was carried out as a case study at the computer laboratory of a primary school in Athens with four 6th grade (last grade of the primary level in Greece) students divided in two groups (G1 and G2) consisted of one boy and one girl. These pupils had already been introduced to the notion of fraction in the traditional classroom. Each pair of students was assigned to one computer. Four 90-minute meetings were conducted by each group of pupils. A team of two researchers participated in each data collection session using one camera and two tape-recorders. All that was said in each group was captured by one tape-recorder. One researcher

was occasionally moving the camera to both groups to capture the overall activity and other significant details in pupils' work as they occurred. The researchers occasionally intervened to ask the students to elaborate on their thinking with no intention of guiding them towards some activity or solution. Verbatim transcriptions of all audio-recordings were made. For the analysis, we adopted a generative stance (Goetz & LeCompte, 1984) allowing for the data to shape the structure of the results and the clarification of the research issues. The identified illustrative episodes can be defined as moments in time which have particular and characteristic bearing on the pupil's interaction with the available tools accompanied with the constructed mathematical meanings.

RESULTS

Fractions as points and segments on the number line

In their efforts to conceptualize certain aspects of the tasks, pupils have constructed their own pictorial representations of the number line as a means to visualize the contextual information they associated with the notion of fraction on the number line (Problem 1). We observed that this kind of activity appeared only at the beginning of the experiment, highlighting the students' attempts to link their intuitions about fractions, the representation of numbers provided by the tool and the conceptualisation of the given tasks.

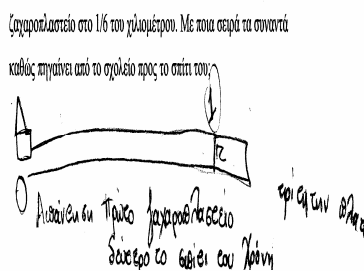


Figure 2: Individual representation of the number line (Problem 1a).

As far as the individual representations became part of pupils' activity, their interactions with the computer environment became strongly associated with the respective points on the screen. Pupils started to speak of them as if they were points, though, from a mathematical point of view, are measures of distances. According to the language of the problem at hand, students continued to refer to the numbers on the FM number line either as points or segments. For instance, when they were calculating the position of certain places they were showing points, while when they

were calculating certain distances, they were showing parts of the number line.

Kinesthetic and conceptual aspects of ordering fractions

Another aspect of the student's conceptualisation of fractions concerned the emergence of body-syntonic activities in ordering fractions by value. The use of the tool's functionalities at that time was interwoven not only with the available representational infrastructure of the tool but also with the task design. This was achieved by students' interpretation of the feedback concerning the position of fractions on the number line as feedback concerning the representation of a distance in an everyday context. This particular idea, in turn, facilitated the integration of the

provided representation of fractions with pupils ‘walking’ on the number line to reach the places mentioned in the given problems.

S1: [*Showing with her figure the distances on the number line of the screen*] So, it is twice this distance. She went from home to school, came back, went to playground and then came back home again.

S2: [*Trying to confirm*] She went to school, came back and from home ... to playground.

In these lines, students (Group 1) display an attempt at making sense of the total distance covered by Constantina as described in Problem 3. As their utterances reveal, the conceptual realm remains subjected to the kinesthetic experience by the metaphor that the child “walks along” the number line indicating points (places) and distances (segments). In the same group this kind of activity lead to the elaboration of ordering fractions based on the comparison of the directed position of certain points from zero (i.e. from the left to the right) in terms of the unit distance on the number line. In trying to find some fraction for the position of Lazaros’ house, situated between the super market and the playground (Problem 2b), pupils were engaged in experimenting with the tool to find out some fractions between $\frac{2}{5}$ and $\frac{3}{4}$. The following excerpt highlights students’ artifact-mediated activity regarding the role of the denominator in the value of fractions.

R: How did you manage to find out this fraction [*i.e. $\frac{3}{5}$*]?

S1: We played with the denominators. For instance, the fraction $\frac{3}{4}$. We chose a bigger denominator, which decreases the fraction, and we estimated that it will fall in between.

R: Did you increase the numerator also?

S1: Yes. Let’s try one.

There is a dynamic aspect in ‘playing’ with the values of denominator and numerator as it is expressed here by S1. The plurality in keeping up the partitioning process seems to play a leading role in conceptualising the order of fractions as well as the nature of the ongoing partitioning itself. Based on the coordination between their existing knowledge (i.e. the value of a fraction decreases when the denominator increases) and the computer feedback, both groups of students explored the continuity of the number line and constructed meanings related to the notion of infinity of fractions that exist between two fractional numbers. In the following excerpt, for instance, students (Group 2) had already constructed twelve fractions between $\frac{2}{5}$ and $\frac{3}{4}$ following the above method and appear ready to continue.

R: So, there are surely 12 fractions Do you think that you could find more?

S3, S4 [*Together*]: Yes.

S3: There could be made many. We can fill all this, from here up to there [*Shows in the computer screen the segment in the number line between $\frac{2}{5}$ and $\frac{3}{4}$*], with fractions.

In this context pupils used the tool to verify their conjectures on the exact position of certain fractions on the line or to check the results of their calculations in paper and pencil. However, due to specific functionalities of the tool both groups had difficulties in connecting the results of certain operations with the solution to Problem 5 which needed the addition of integral and fractional parts of the number line. This was mainly related to the fact that the symbolic representation of a calculation in the FM includes only the numbers of the respective operation (e.g. $1/3 + 1/2$) and not the resulting equivalent fraction (e.g. $5/6$). Moreover, the user is not able to measure a specific part of the number line (e.g. between $1/2$ and $3/4$) if it is not starts from zero. So, the measure of a specific segment on the number line is realized by its equivalent transformation starting from zero. Both groups of students did not succeed in coordinating the conceptual part of the artifact-mediated operations that recognised as necessary for the solution with the representation of the respective fractional amounts on the number line. Considering Constantina's house on zero, for instance, students (Group 1) found that Efi's house is on the point $5/3$, while the playground is on the point $3/4$. In solving Problem 5, S1 needed to compare $3/4$ with the result of the subtraction $5/3 - 3/4$.

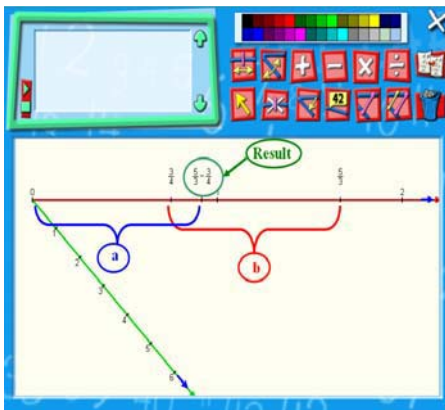


Figure 3: Two interpretations of the subtraction $5/3 - 3/4$.

R: How did you find that point [shows the point labeled as $5/3 - 3/4$ on the number line]?

S1: I subtracted $5/3 - 3/4$. Because $5/3$ is the whole and I subtracted the part from zero up to the playground. So, we've finished.

R: Ok. Can you answer now who walks longer?

After calculating the result of the above subtraction ($11/12$) in the paper, S1 answers:

S1: Eleven twelfths.

Though completed the requested calculations concerning the distance 'b' (Figure 3) and transformed it to the (equal) segment starting from zero (i.e. the distance 'a' represented by the expression $5/3 - 3/4$ on the screen), S1 is unable to connect it with the paper and pencil difference consisted of one fractional number ($11/12$). The difficulty in distinguishing the equality of the above expressions is obviously related to the limitation of the educational exploitation of the representations and functionalities of the FM in the present study.

Abstracting the position and the size of the numerical unit

One of the most critical conceptual jumps that contribute to the children's difficulty in learning fractions concerns the role of the unit. In number-measure situations particularly, this problem seems to arise from the fact that children are unable to decide what constitutes an appropriate 'unit' on the number line, since may be induced to taking any whole line segment shown as representing a unit, rather than

the line segment between two successive integers. The unit is thus at the core of the main mechanisms that play a major role in naming distances from zero as well as in understanding the relation between the size of the unit and the location of specific points on the number line. In our experiment students' conceptualization of unit appeared in different phases of their explorations as an indication of their progressive familiarisation with the control of the mathematical nature concerning the construction and representation of fractions in the FM. In several cases distinguishing that the numerical unit of measure was regardless of its position on the number line was facilitated by the preceding kinesthetic interpretation of the proposed tasks. In the next excerpt students (Group 2) integrated the (unit) distance between Constantina's house (corresponding to 0) and the school (corresponding to 1) in their approach and used it to bypass the constrain of representing numbers on the left of the zero point of the FM number line. After experimenting with different positions of Efi's house in relation to Constantina's house and to school (Problem 3), S3 tries to explain to S4 the possibility to consider Efi's house on the left of the point 0. In doing so, he also indicates that the distances between different places (i.e. points) are independent from the position of the unit on the number line.

S4: Where is Constantina's house? We know that it is 1 kilometer far from school.

S3: At 0.

S4: How do we know that?

S3: We symbolised it here (*He shows the distance from 0 to 1*). We can do the same with 3 and 4 or 9 and 10. We just preferred 0 and 1 for school. Do you understand?

It is noticeable in the above excerpt that the indexical gestures of S3 on the number line appear as part of the situated abstraction concerning the position of the unit, to indicate in a precise way its imagined different positions on the number line. In a similar way, students abstracted also the independency of the length of the numerical unit from the fractional parts of it since, once determined, it remains so throughout a problem. This was mainly achieved through the extensive use of the scaling functionality of the FM by which students could dynamically change the measure of the numerical unit.

R: Why do you enlarge it?

S3: Because they were stuck together and they seemed as if they were 67.

R: Yes, they are more discernible, but are they the same? Does anything change?

S3: No, just to discern them.

Triggered by the need to zoom on specific parts of the number line, S3 enlarges the numerical unit indicating a suitable understanding of the fractions-as-measures used to determine distances on the number line. This also implies the conceptualisation of ratio scales which require both equal intervals and an absolute zero.

CONCLUSIONS

Dickson et al. (1984) pointed out that many children are likely to be uncertain about the nature of a fraction as a number-measure represented by a point on the number line well into the secondary age range. In this study however, some interesting meanings around the measure personality of fractions seemed to have emerged while 12-year-olds were using the symbolic, graphical and manipulation tools of the FM s/w. Working within these tools, the dual representation of fractions as numbers and points on the number line, the order of fractions and the role of the unit came into play offering us an interesting terrain in which to investigate the nature of pupil's engagement in meaningful experimentation with the fractional amounts of their activities to measure distances on the number line and compare them in mathematically efficient ways. The foregoing episodes illustrate the pupil's progressive focusing on connections between the measure interpretation of fractions to the other concepts, situations and representations of the relative conceptual field. In the first part of the analysis an icon-driven individual representation of the number line seemed to mediate pupils' familiarisation with the representations provided by the tool. The second part of the analysis showed clearer how the proposed tasks provided a context for the students to coordinate the interplay between the points as static numbers to the distances as dynamic measures from zero. In these cases, the mathematisation of pupils' responses in ordering fractions was inextricably related to the kinesthetic nature of the computer feedback translated in the context of the given activities. Students' inability to conceptualise the equality of fractional expressions and its representations provided by the tool signalled the development of an interesting domain to extend the educational potential of the software in a future extension of the present study. In the last part of the results pupils' previous experience with the FM tools had been moving in the direction of scaling the numerical unit on the number line by abstracting both its position and its size. It is thus suggested that the dynamic access to static representations provided by digital media, like the FM, could possibly (a) change our conceptions about what can be learnable at school, in what way and at which grade and (b) indicate a need to reconceptualise the existing curricula with the integration of technology.

NOTES

1. The research took place in the frame of TELMA (Technology Enhanced Learning in Mathematics), a European Research Team (ERT) established to focus on the improvements and changes that technology can bring to teaching and learning activities in mathematics. TELMA is part of Kaleidoscope, a Network of Excellence funded by the European Community (IST-507838).
2. The FM is part of AriLab2, a stand-alone version of an open system developed by the Consiglio Nazionale delle Ricerche - Istituto Tecnologie Didattiche (CNR-ITD) research team in Genoa (Italy). AriLab2 is composed of several interconnected microworlds based on the idea of integrated multiple representations and functionalities designed to support activities in arithmetic problem solving and in the introduction to algebra.

REFERENCES

- Aksu, M. (1997). Student performance in dealing with fractions, *Journal of Educational Research* 90(6), 375-380.
- Baturo, A. & Cooper, T. (1999). Fractions, reunification and the number-line, *Proceedings of the 23rd PME Conference*, Haifa, Israel, Vol. 2, 81-88.
- Charalambous, C. & Pitta-Pantazi, D. (2005). (in greek) The role of number line in the evaluation of the notion of fraction as number-measure, C. Kynigos (ed.) *Proceedings of the First Conference of the Greek Association of Research in Mathematics Education*, University of Athens, Greece, 404-414.
- Dickson, L., Brown, M. & Gibson, O. (1984). *Children Learning Mathematics: A Teacher's Guide to Recent Research*, London, Cassel Educational Ltd.
- Goetz, J. P. & LeCompte, M. D. (1984). *Ethnography and Qualitative Design in Educational Research*, Academic Press, London.
- Hannula, M.S. (2003). Locating fraction on a number line, *Proceedings of the 2003 Joint Meeting of the PME and PMENA*, University of Hawaii, Vol. 3, 17-24.
- Harel, I. and Papert, S. (1991). *Constructionism*, US: Ablex Publishing Corporation.
- Kynigos C. (2002). Generating cultures for mathematical microworld development in a multi-organisational context, *Journal of Educational Computing Research*, Baywood Publishing Co. Inc. (1 and 2), 183-209.
- Lamon, S. J. (1999). *Teaching Fractions and Ratios for Understanding*, NJ: Lawrence Erlbaum Associates.
- Marshall, S.P. (1993). Assessment of Rational Number Understanding: A Schema-Based Approach, in T. P. Carpenter, E. Fennema, T. A. Romberg (Eds.), *Rational Numbers: An Integration of Research*, Lawrence Erlbaum Associates, 261-288.
- Ni, Y. (2001). Semantic domains of rational numbers and the acquisition of fraction equivalence, *Contemporary Educational Psychology*, 26, 400-417.
- Noss, R. and Hoyles, C. (1996). *Windows on Mathematical Meanings*, Kluwer.
- Siegel, M., & Smith, J.A. (1997). Toward making representation count in children's conceptions of fractions, *Contemporary Educational Psychology*, 22, 1-22.
- Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fraction, *Learning and Instruction*, 14(5), 503-518.
- Vergnaud, G. (1991). La théorie des champs conceptuels, *Rechercher en Didactique des Mathématiques*, 10(2/3), 133-169.

TO SENSE AND TO VISUALIZE FUNCTIONS: THE CASE OF GRAPH STRETCHING

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In this paper we observe the very first attempts of two calculus students to understand the concept of dilation of functions using tools that support direct manipulations of graphs. Each student was engaged in interpreting dynamic graphs in order to construct a procedure for dealing with graph stretching in various situations. There is a similar conceptual metaphor at the core of each procedure: stretching and straightening a rope. We are seeking the roots of this metaphor and its role in the process of constructing meaning for this concept based on theories of embodied cognition and on theories attempting to explain the instrumentation of artefacts.

THEORETICAL FRAMEWORK AND GOALS

Studies suggest that developing "function sense" with regard to function properties and operations is a central task of the mathematics curriculum (Eisenberg and Dreyfus, 1994). Developing competence in solving function-related problems means being proficient in multiple linked representations of functions (graphic, algebraic, and tabular) and learning to move freely between them. Yerushalmy & Schwartz (1993) demonstrated that the symbolic representation of a function reveals its process nature, whereas the graph suppresses the process nature of the function and invites encapsulation. Traditionally, transformations have been taught with a strong emphasis on algebraic symbolism, with emphasis on the rote learning of algebraic rules (Borba and Confrey, 1996). Geometric transformations of functions and direct manipulations of function graphs treat functions as on-screen mathematical objects. Thus, understanding transformations presupposes a good understanding of functions as an object (Eisenberg and Dreyfus, 1994).

Dilations (sometimes called stretching) preserve the overall shape of the graph but change the distance of each point to an anchor line by a fixed ratio k . $f(x) \rightarrow f(kx)$ is a horizontal dilation and $f(x) \rightarrow kf(x)$ is a vertical dilation.

Transformations in the horizontal direction were found to be less well understood than those in the vertical direction (Eisenberg and Dreyfus, 1994, Zazkis, Liljedahl and Gadowsky, 2003). Students find it easy to visualize the translation of a function by moving its graph rigidly, but have difficulty finding a proper image for stretching even a simple graph.

Various technological tools have been designed to support the effective learning of functions (see Kieran & Yerushalmy 2003 for an extensive review of such tools). The tools support the *immediate visualization of multiple linked representations*. Any action carried out on a specific representation provides immediate change and feedback in all representations, and the tools enable *direct and sensual manipulations of graphs*. Expressions that have traditionally been considered as the ultimate way of parameterizing a member into a family are not longer the only entry.

Bruner proposed a model of how children construct knowledge through three hierarchical modes of engagement with representations: *enactive* (manipulating concrete objects), *iconic* (visual and sensory recognition), and *symbolic* (abstract reasoning) (Bruner, 1966). The first and essential step is that of enactive, sensory-motor actions. Recent research of embodied cognition emphasizes that the major source of many abstract mathematical ideas is in concrete sensory-motor experiences mediated by metaphoric thinking (Lakoff & Núñez, 1997). Based on Bruner's theory and considering the embodied cognition theory, Tall (2003) identified three worlds of mathematics thinking: *embodied, symbolic-proceptual, and formal-axiomatic*. The first stage of our study focused on the metaphorical thinking demonstrated by the students' sensory-motor experiences with geometric transformations. We therefore describe the construction of knowledge that takes place in the embodied world (in

Tall's sense). We assume that mathematics is permanently created and sustained by imaginative resource such as conceptual metaphors, which serve as a mapping between a well-known source domain, accessible through perceptual experience, and an un-known target domain, which is the world of mathematical entities (Sfard, 1997). After this stage our students were challenged with activities that required a transition to the symbolic world. The symbolic-proceptual stage is not within the scope of the current paper. Here we examine the interplay between the wide bodily experience and the act of stretching in the real world. We study the perceptual experiences acquired through direct manipulations of graphs using the technological tools in order to construct meaning for the geometric transformation of functions.

METHOD

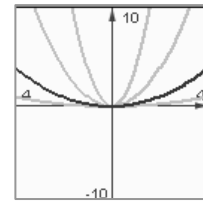
The present paper is part of a broader research in which we interviewed five calculus students in four task-based interviews. We assigned tasks to promote inquiry into the concept of linear transformations of functions with computer applications that are part of the Function eBook (Yerushalmy, Shternberg & Katriel 2003). The various tools of the eBook enable direct transformations of graphs using dragging or "arrows" (control buttons for executing specified discrete steps that hint to the direction of the transformation). The direct actions that can be performed on the graphs were designed to provide opportunities for acting at the senso-motor level and to strengthen visualization. We describe the work performed by two 11th grade students, Gili and Shawn, during their first interview, which focused on the embodied world. Their task was to describe whether and how a given function can be translated or stretched to fit a second given function.

FINDINGS

We will describe how Gili's initial sensory image was supported by the interaction with the tool, how Gili needed to adjust and modify her image to the tool view and how the tool itself stimulated Shawn's new sensory image.

1. Strengthening the initial sensory image

Gili experimented with the tool's "stretching by dragging" feature for the first time. She stretched a parabola with a vertex at the origin by holding one of the branches with the mouse and dragging it out. She expresses her impression using the words "stretch out," "I tried to pull it outside..."), and using a gesture of clenching her fists as if she were holding something and pulling it out fiercely in both directions.



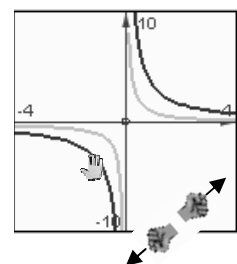
Interviewer: So what did the stretching do?

Gili: Like it pulled them... straighten them.



Although Gili performed the stretching by dragging a single point on the graph, her gesture shows her holding the two branches of the parabola and bending it until it becomes a straight line. The dragging action and the virtual motion on the screen may have caused Gili to imagine the stretching as pulling a curved rope. Indeed, "stretching" can be interpreted in several ways in everyday terms. To stretch is to make longer or wider by pulling. One can stretch a spring or a rope, and one can stretch one's body. In Hebrew, the word "stretch" is used also in the meaning of "drawing a line" between two points. We wondered why Gili initiated this physical image and whether and how the image would serve her in other situations.

On her next task, Gili had to find out how a given hyperbola was transformed to create the given picture. She answered immediately "like this," clenching her fists again and pulling them in opposite directions. She repeated the earlier procedure of catching the



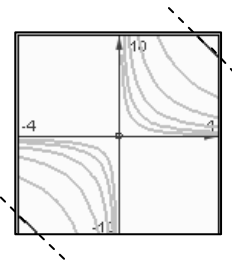
"vertex" of the hyperbola with the mouse and pulling it out. Gili obtained the expected visual image because the dragging tool enables simultaneous stretching in both the vertical and horizontal directions. To reinforce her assumption, she continued dragging the curve, expecting it to become a straight line.

Gili: O.k. what is it going to do now? ...(1) I don't know what it is doing... maybe it's going that way...(2)
That it's two straight lines. It seems that this is the trend.

(2) One more stretching with the "arrows" and the graph disappeared.
(3) She raises her hands pointing at two imaginary parallel lines.

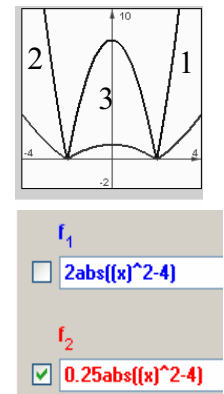


To see the trend, Gili turned to the "arrow" buttons that allowed her to control the stretch in discrete steps. The systematic change on screen inspired her again to use the metaphor of pulling and straightening a rope inside the frame of the screen.



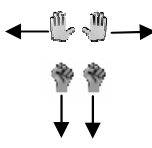
2. Modifying the sensory image

The next task required fitting two absolute value quadratic functions using transformations. Gili visualized the curve as a parabola consisting of three parts: two branches (like a parabola) and a "dome" in the center. She assumed that stretching was applied but was not able to identify how it was done. She expected that grabbing the outer branches and pulling them outward would not change the "dome" similarly to the unchanging center parts of the parabola and the hyperbola earlier.



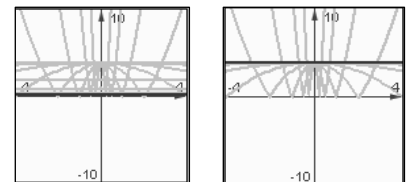
Gili: Oh, This is really strange. It doesn't make sense to me. (1)
...Because if I move these (2), just the sides down, it doesn't make sense to me that this "dome" (3) will be stretched too.
I feel that if I stretch something (4) than it goes sideways.
And here (5), it is just being lowered. I don't know... Here it goes sideways. (6)
And this looks like it is open and here it cannot be open so it goes downwards. (7)

(1) Clenches her fists and bends them..
(2) Points to the outer branches
(3) Points to the dome
(4) Moves her fists outward.
(5) Points to the dome.
(6) Opens her hands
(7) Pulls down



Gili expressed her feeling [4] that the curve should go sideways but would not change in the center. She ran a mental simulation of her dominant sensorimotor metaphor of pulling a rope, and expected the length of the curve to be retained, as in the case of physical rope. Trying to make sense of what she saw, Gili had to modify her metaphorical image and make it consistent with the tool perspective. She accepted the fact that the center moved downward as a constraint [7].

Having accepted the tool image of the dome stretching, she tried to straighten the curve to create a horizontal line coinciding with the x-axis by using the dragging stretch tool. Among the series of transformed functions she obtained two straight horizontal lines: a horizontal line through the maximum point and an additional line coinciding with the x-axis. In fact, Gili did straighten the graph but contrary to her image she obtained two possible lines. The need to settle this contradiction led her to another modification of her visual metaphor.




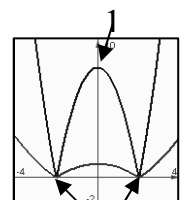
Gili: Ahha... that is either I stand here, (1) put my finger here and stretch...(2) or I put the fingers...(3)
 Ahha, aha, either in the section with the x-axis, (3) or in section with the y-axis.
 I can put my finger and stretch.
 And these would be the two horizontal lines that could be stretched.
 I think so.


Interviewer: In which direction can you stretch?

Gili: Only to the side.

Looks at the graphs.

- (1) Places a finger on the maximum point.
- (2) Moves her fists outward. 
- (3) Places two fingers on the two minimum points.



Points right and left and pulls. 

Gili needed to perform a vertical stretching, but she still was not aware that there were two stretch directions, horizontal and vertical. To reconcile her stretching

image with the picture on the screen she fixed the special points making them invariant, and the rest of the curve stretched to the sides. The dragging tool enabled her to confirm and develop her visual image of straightening the curve, although it allowed her to disregard the direction of the stretch.

3. Tool-stimulated *new* sensory image

Shawn was a strong math student, verbal, and cooperative. During the interview he tended to explain the phenomena he observed using algebraic methods (e.g., stretching a parabola meant changing the coefficient of the x^2). Shawn preferred to stretch in discrete steps using the "arrows" rather than by direct stretching. This tool provided him with a static picture of the starting and final states of each stretch.

Analyzing Shawn's discourse reveals that, similarly to Gili, he used the "stretching a rope" metaphor, but in his own way. He did not grab the branches of the graph but imagined taking hold of the graph in its special points (e.g., maximum and minimum) and moving these points to their final state while the rest of the graph points followed along. Shawn experienced some stretching and tried to describe the stretch in the vertical direction:

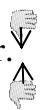
Shawn: ... and as far as I stretch ...(1)
 I see, this action (1) does this..(2)

 It actually transforms it (3) to less extreme.
 That is, all the lines become less... convex.

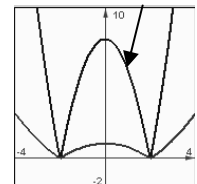
(1) Points to the button stretching downward.



(2) Brings his hand closer.



(3) Points to the upper graph.

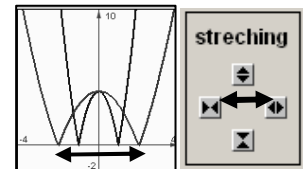


Shawn compressed the graph using the stretch down button and pointed at it to describe his action. The relative location and shape of this button hints at the nature of the stretch. He demonstrated the graph change using a gesture of closing his two hands in a way that imitated the arrows icon. He did not simply move the maximum points to make them coincide but rather shrank the entire graph. The design of the

buttons and their location, together with his gestures led him to characterize the nature of the stretch as changing the curvature of the graph.

Shawn: First I must press on the stretchiness to the right and to the left (1) for the axes to be at the same place... (2).
It's gonna' be alright.

(1) Points to the right and left horizontal stretching.
(2) Points to right and left minimum points.



Again Shawn worked with the arrows, which forced him to choose in which direction he should stretch (up/down or left/right). He decided to stretch it out horizontally to the right. The action stimulated a sensory memory of stretching from his perceptual experience.

Shawn: ... it means it stretches them...like a spring...(smiling)

Bends his palms one against the other and pushes out.



Shawn surprised himself in his spontaneous association of stretching a spring to describe the mathematical reality of the tool, and expressed doubt about it.

Shawn: ... but as a matter of fact, it doesn't lower either..(1)
So it is not exactly a movement like (2)...a spring ...ahh, stretch...

(1) Points to the middle of the graph.
(2) Closes his thumb with his index finger and pulls outward.



Interviewer: I don't understand how it is different.

Shawn ... because of... we did here a stretch... and that length is smaller than this length..

Shawn ran a mental simulation, as had Gili. He used the metaphor of a spring being stretched to describe stretching with the tool. But contrary to Gili, who adjusted her visual metaphor to the tool view, Shawn claimed that there was no exact isomorphism between stretching in real life and with the tool because it contradicted

the assumption of length preservation. Although the visual metaphor was an integral part of his reasoning, he insisted on making a clear distinction between real life and the mathematics viewed produced by the tool.

DISCUSSION

The students were thinking with and through language, their bodies, and the artefacts. The construction of meaning was amply sustained by natural language. The students used the "rope stretching" metaphor interpreting the Hebrew word "stretch" literally, and used their senso-motoric memory for this action. (One wonders what metaphor would be used if the activity introduced the word dilation, which brings to mind the pupil of the eye.) The students frequently used gestures that reflected the visual metaphor, and even influenced it. Moreover, we maintain that the tool they used had a crucial role in stimulating the visual metaphor and responding to it. First, because of the *immediate and dynamic images* of the transforming graphs, students viewed these graphs as manipulated objects. The dynamic pictures were the perceptual and visual resources used to run off-line simulation of stretching by means of a sensual metaphor. Second, the *direct and sensual manipulations of graphs* enriched the students' perceptual experience that prompted the visual image. The tool aroused an on-line sensory stimulus through which the students could act in a tangible and concrete way on the abstract functions. The personal experience with the tools produced a personal style of constructing a meaning for the concept. Gili preferred to work with the dragging tool, describing the stretching as dynamic and continuous process. Shawn worked with the arrows, which caused him to see the stretching as transforming the function from an initial to a final state. Using the arrows, Shawn paid attention to the direction of stretching, while Gili overlooked it.

Based on learning theories (Bruner, 1966, Tall, 2003), the sensory metaphoric image that was guiding the construction of meaning through interaction with the tool should or could be the foundation for the construction of sign and symbol manipulations. It is an important challenge to identify and analyze these transitions.

REFERENCES

- Borba, M. C. & Confrey, J. (1996). A students' construction of transformations of functions in a multi representational environment. *Educational Studies in Mathematics*, 31(3), 319–337
- Bruner, J. S. (1966). *Towards a Theory of Instruction*, New York: Norton.
- Eisenberg, T., & Dreyfus, T. (1994). On understanding how students learn to visualize function transformations. In: E. Dubinsky, A. Schoenfeld & J. Kaput (Eds.), *Research in collegiate mathematics education* (Vol. 1, pp. 45–68). Providence, RI: American Mathematical Society
- Kieran, C. & Yerushalmy, M. (2004), Research on the Role of Technological Environments in Algebra Learning and Teaching. In: K. Stacey & H. Chick (Eds.), *The Future of the Teaching and Learning of Algebra: an ICMI study*. Dordrecht, The Netherlands: Kluwer.
- Lakoff, G., & R. Núñez (1997). The metaphorical structure of mathematics: Sketching out cognitive foundations for a mind-based mathematics. In L. English (ed.), *Mathematical Reasoning: Analogies, Metaphors, and Images*. Mahwah, N.J.: Erlbaum.
- Radford, L. (2005). Why do gestures matter? Gestures as semiotic means of Objectification. In Helen L. Chick, Jill L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, University of Melbourne, Australia, Vol. 1, pp. 143-145.
- Sfard, A. (1997): Commentary: On metaphorical roots of conceptual growth. In: L. English (ed.), *Mathematical Reasoning: Analogies, metaphors, and images*, 339-371. Hillsdale, NJ: LEA.
- Tall, D. (2003). Using Technology to Support an Embodied Approach to Learning Concepts in Mathematics. In *L.M. Carvalho and L.C. Guimarães História Tecnologia no Ensino da Matemática*, vol. 1, pp. 1-28, Rio de Janeiro, Brasil
- Yerushalmy, M., Katriel, H. and Shternberg, B. (2002) *The Function Web-book*. www.cet.ac.il/math/function/english
- Yerushalmy, M. & Schwartz, J. L. (1993). Seizing the Opportunity to Make Algebra Mathematically and Pedagogically Interesting. In Romberg, T. A., Fennema, E., and Carpenter, T. P. (Eds.), *Integrating Research on the Graphical Representation of Functions* (pp. 41–68). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Zazkis, R. , Liljedahl, P. & Gadowsky, K. (2003). Conceptions of function translation: obstacles, intuitions, and rerouting. *Journal of Mathematical Behavior* 22 , pp. 437–450

TOOLS THAT FORCE REFLECTION

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ABSTRACT

This paper describes how the implementation of a well thought out environment can help children construct and reflect on understandings of randomness. The paper reports a part of the findings of a study in which children aged 11 to 12 were actively engaged in playing and creating chance games in a game-playing computer based environment in which each child had three weeks to play six games, finish two unfinished games and create his own chance game. The episodes from the case study presented indicate that in such environments the children can be engaged with the environment to a degree that allows initial resources to be explored and reshaped while emergent sense making is constructed and acquires priority in the children's network of understandings.

Background

Over the years a considerable effort has been made in researching children's probabilistic reasoning (Piaget and Inhelder, 1975; Hawkins and Kapadia, 1984; Fischbein & Gazit, 1984; Tversky and Kahneman, 1972; LeCourte, 1992; Konold, 1989). Nonetheless, in much of the research the setting was not considered or examined as having a significant influence on children's construction of probabilistic knowledge and the activities used were restricted to instructing knowledge or identifying strategies rather than allowing construction and exploration from the child's point of view.

Carraher, Carraher and Schliemann (1985), Lave (1988) pointed out the importance of the structuring resources in a setting; and Wertsch (1998) in his premise of *mediated action* suggested focusing on agents and their cultural tools- the mediators of action. While Wertsch's (1998) perspective places significance on the affordances provided by the cultural tool, and he suggests studying agents and tools as they interact, other researchers emphasise that computational environments can facilitate rich and multiple representations of mathematical concepts (Noss and Hoyles, 1996; Wilensky, 1993; Pratt, 1998, 2000; Pratt and Noss, 2002). In Wertsch's (1998) terms a computational environment, as a cultural tool, can act as a mediator of action and therefore as a mediator for the construction and development of mathematical concepts.

Constructionism's use of 'building', 'constructing' or 'knowledge representing' as central metaphors for a new elaboration of the old idea of learning by doing rather than by being told (Harel and Papert, 1991, p. 1) prepared the ground for giving the opportunities to children to programme and run their own representations of mathematical concepts (Kahn, 1999; Papert, 1980; Goldstein & Pratt, 2001; Goldstein et al., 2001). In line with this view, two hypotheses initiated this research:

- a) by designing tools which provide resources not available in conventional everyday experience and give learners the right kind of concrete and meaningful experiences, this might make it possible to observe building of solid probabilistic intuitions and new internal resources (Pratt, 2000; Wilensky, 1993) and
- b) if children have the opportunity to be in charge of their own learning in a creative environment and are given the chance to identify, question and test the limits of their own intuitive beliefs, then they will develop behaviour that would not be predicted by the existing literature.

The design approach

The aim of this paper is to describe the design of an environment which

- a. Affords expressive power to children aged 11 to 12 in the domain of probability.
- b. Enables children to develop *purpose* when applying probability, become engaged with random events and develop a sense of their potential and limitations, while at the same time coming to appreciate the *utility* of the concept (Pratt et al, 2006).
- c. Captures children's initial resources as they arise from the children's preliminary dealings with the environment and their emerging sense making as they interact and make connections to it.

In order to bring the design aims into view, the ToonTalk microworld was chosen as a user-friendly environment in which children could play and build chance games. ToonTalk is an animated interactive world where children build and run programs not by typing text or arranging icons but by performing actions upon concrete objects (Kahn 1999). To begin with, the ToonTalk world resembles a modern city in which programming can be done from inside a virtual world. The programmer acts as a ToonTalk character which starts off flying his/ her helicopter over a city. After landing he/ she enter one of the houses with a toolbox which contains tools to build and run programs. When the programmer enters a house the toolbox opens up to reveal basic elements like a number, text and pictures as well as more complicated tools like a bird (a way to send messages, channel transmit capability), a nest (a way to receive messages, channel receive capability), a robot or clause (the smallest coherent program fragment), a box (a container of items), a set of scales (which tests if something is more than something else), a loaded truck (process generator) and a bomb (a process termination tool), all of which can be altered to suit the needs of the programmer.

The idea in creating the microworld in ToonTalk was to create an *autoexpressive* computational environment as suggested by Noss et al (1997); that is 'an environment in which the only way to manipulate and reconstruct objects is to express explicitly the relationships between them' (p.207). Therefore, the tools which were created aimed to afford ways of linking seeing, doing and expressing (the prepared games and the unfinished games) while at the same time all these links would be tested when the children manipulated the tools in order to create their own games.

Although basic tools for representing random situations were already created within the ToonTalk environment, in carrying out this project there was a need to create tools that would be easily manipulated and represented in various ways in the games. Therefore, the sample space was represented by a garden (covering both replacement and non-replacement situations) on which objects could be placed, and an outcome tool was added which the children could link to a particular garden and get an outcome when pressing a letter (see for example figure 1). Because these two tools were linked and one was directly affected by the other, the mechanism behind them was shown to the children after constructing some primary understandings of their behaviour when playing games.

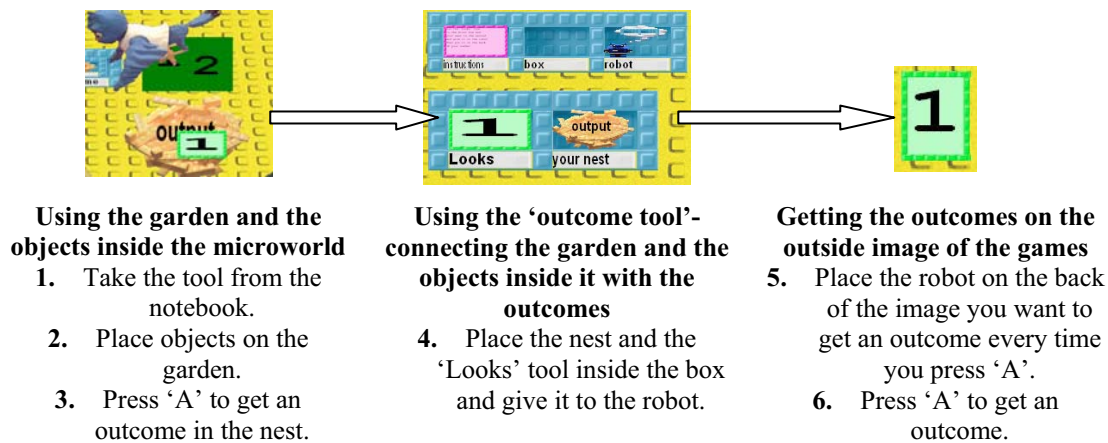


Figure 1: Tools created within the ToonTalk microworld

The principle behind the tools created in this project followed Hoyles and Healy (1997) as well as Harel and Papert's (1991) approach of the constructionist perspective in computer design where 'learning by making' is taking place; and the created 'tools will have to do just enough to illuminate structures and relationships without solving the task completely', in order to direct their usage in 'an explicit appreciation of the form of generalised relations while functionality and semantics are preserved and expended' (Hoyles and Healy, 1997, p.28).

The methodological approach

To enable a detailed exploration of ideas and sense making, ten case studies of children age 11 to 12 were conducted and video-taped. The project took place in a school in Cyprus over a period of two months when the students were engaged with a specially design computer microworld in order to play, finish and create chance video-games. A qualitative research strategy was employed, in which, in addition to the video-recording process, data from observation, task interviewing, focus sessions, a diary for children to keep and all the computer interactions, as well as the different stages of the game and the final game were all collected.

The project was divided between three phases in order to provide opportunities for the children to play and build games by gradually introducing them to the tools created in the computer-based environment. During the first and the second phase the

children worked in pairs to play six chance games and to finish two unfinished games. In the third phase, children built their own games.

Tools that force reflection through a case study

While the children were playing, finishing and building games three main categories of understandings were identified: *personal understandings*, *inductive images* and *mathematical understandings* (figure 2). At this point I will briefly describe the theoretical framework before demonstrating it through a case study.

A ***personal understanding (P)*** in probability is a well-established mental resource that is built up by connections which are supported by deterministic and causal factors; it is empowered by the everyday (in-school and out-of-school) encounters with chance; and is surfaced by the immediate appearance of the computational device or by a relatively short-term usage of the device. It is shaped by the child's intuitions and beliefs about random phenomena and it has relatively little relatedness to formal probability.

The term 'personal understandings' came out of Pratt's (2000) position in which he gives significance to local resources, and it aimed to emphasize an internalised resource that children brought to the activity. It is personal because of the way that it is constructed, through personal experiences with probability and chance situations. It refers to initial understandings which are stabilized in the child's sense making through their previous experiences with randomness and therefore the meanings they assign to phenomena are constructed from partial interactions with the concept.

An ***Inductive Image (II)***: is a sense-making resource which emerges through activity and is articulated in specific situated terms. An inductive image acts as a resource based on a set of new connections that are made by the child and are situated within the features of the computational environment. When having an inductive image the child and the computer environment are engaged in a dynamic relationship in which they are influenced by each other and through this interaction the child is establishing interconnections within the medium.

The term 'inductive image' is formed to include evolving sense making within the setting which refers to discovering, developing and applying a general principle from a set of facts and observations of the tools available. It incorporates Noss and Hoyles' (1996) situated abstraction as it refers to a general view of the stochastic phenomena drawn directly from the situation which allows learners to reflect on the structures within the setting and make sense of phenomena.

The inductive images identified in this study have a developmental character and that comes as no surprise if we consider the fact that first attempts by the child to make sense of the environment were in terms of his/ her personal understandings and these understandings are based on links to deterministic and causal factors. Therefore, the growth of inductive images starts from these links and takes on new connections to the environment that the children are interacting with.

In this study six inductive images were produced and developed: the ‘more of’, ‘equally likely outcomes’, ‘position of the objects’, ‘simple garden’, ‘non-replacement garden’ and ‘fair garden’. The inductive images of ‘more of’ and ‘equally likely outcomes’ are examining the setting in terms of favourable and unfavourable cases, while the inductive images of the ‘position of the objects’, ‘simple garden’ and ‘non-replacement garden’ are referring to the properties of the random generator in terms of determining a random event. The ‘fair garden’ inductive image sees the fairness of the random generator in terms of the proportion of the objects inside it and is usually expressed by incorporating the rules of the games.

A **mathematical understanding (M)**: is a mental resource which is developed when the child is discovering a general principle from a set of facts but is applied when the child makes sense of the situation in hand by imposing or applying that general principle. It is an understanding described by general strategies and ideas about chance and sample space. It includes the child coming to see a random situation and the process of articulating meanings of randomness.

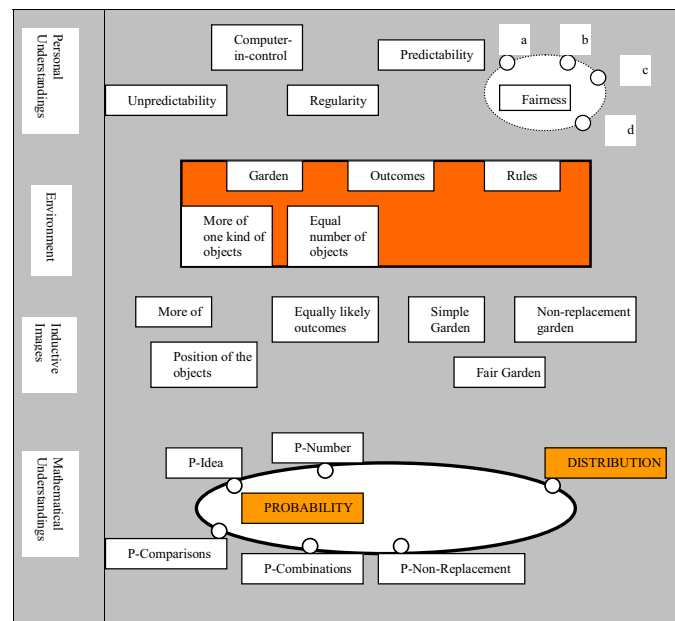


Figure 2: Theoretical Framework

A mathematical understanding represents not only the child coming to see the mathematics embedded into the features of the computational environment but also to make several connections between the interconnections of the mathematics within the medium (fully developed inductive images), the mathematics he learned at school (formulas) and new general strategies that derive from mastering the connections between the previous two. During this study two mathematical understandings were produced and developed: *probability* (the child comes to see the proportion of outcomes for each possibility as predictable) and *distribution* (the observer is able to

exert control over the proportion of outcomes of each possibility through manipulation of the possibility space).

A mathematical understanding goes under the umbrella of Noss and Hoyles' (1996) situated abstraction because, although it is articulated as a general principle (in the eyes of an expert) and holds formal probabilistic reasoning (therefore the term mathematical understanding), it is expressed by the children as a concrete way of reasoning about randomness within the specific environment. The term 'concrete' refers to Wilensky's (1993) theory of concreteness and Noss and Hoyles' (1996) theory of webbing which sees the children being highly involved with multiple representations of a concept.

In the model suggested above personal understandings, inductive images and mathematical understandings shape a network of understandings in which all resources are in constant examination and gain priority through multiple representations and modes of interaction with the concept of probability within the environment. This does not suggest a unidirectional construction of meanings, however. Personal understandings are used alongside mathematical understandings and inductive images and vice versa. While the child is constructing connections within the features of the environment and his/ her sense making resources, this expanding interconnected network of understandings is reshaped and the structure of the network is reorganised to facilitate the necessities of the activity.

At this point I will present some episodes from Christina's case study who according to her teacher is 'a C student [who it would not be] worth wasting research time on'.

Testing initial understandings and constructing the 'more of' inductive image

In the second game of phase A, a non-replacement but visible garden was used of {6 yellow fishes, 5 green fishes}. The children had to guess the next outcome. An imperative difference in this game, that seems to have an impact on Christina's understandings, is that in the second game a destructive garden was used and therefore the attention was shifted into the garden and the proportions of the objects inside it. This is shown by the fact that Christina used a personal understanding in which the frequencies of the objects inside the garden were considered (*computer-in-control*: that the computer has fewer objects to choose from every time they played, lines 021-025, extract 1).

Interview

021. R: is there a difference between this game and the previous one?

022. P: in this game, every time we play a fish is leaving the garden.

023. R: and what does that mean?

024. P: the number of the fishes inside the garden changes....

025. C: yes the computer has fewer objects to choose from.


On screen



P

Extract 1: Christina's task interview, 'Lucky fishing' game

Although her personal understandings, which paid attention solely to the outcomes (*regularity* -she was expecting a pattern in the outcomes; and *unpredictability* -she was holding luck responsible for the outcomes) failed to predict an outcome in the first game, at the beginning of the second game she was still articulating those meanings without apparent discomfort. In a garden of {4 yellow fishes, 5 green fishes} however, Christina's inductive image of '*more of*' (the more frequent an object is in the garden, the bigger the chance of getting that outcome) was developed where she predicted that a green fish is going to come up next because there are more green fishes inside the lake than yellow and it will be easier for the computer to pick it up (lines 050-051, extract 2). Although there was a causal factor hidden in this initial form of the inductive image, this is the first time that Christina made connections between the garden, the proportion of the objects inside it and the outcomes.

Interview	On screen	
047. C: there are 4 yellow and 5 green fish inside the lake.		SI
...		
050. R: what do you think Christina?		
051. C: I think is going to be a green one because we have more green than yellow inside the garden and it will be easier for the computer to pick up a green one.		II

Extract 2: Christina's task interview, 'Lucky fishing' game

Figure 4 (a) demonstrates Christina's understandings during the second game. At this point a system of understandings appears, in which the connections made between the garden, the frequencies of the objects inside it and the outcomes seems to initiate the personal understanding of *computer-in-control* and the inductive image of '*more of*'. Before this inductive image was created, Christina was making estimations based only on the outcomes. Therefore is believed that the inductive image of '*more of*' came up partly because of the nature of the game, the fact that the garden and the consistency of the objects was changing every time they played and partly because of the fact that none of her personal understandings led to a prediction that she could rely on. Furthermore, the personal understanding of *computer-in-control* initiated connections, which perhaps have let to the construction of the inductive image by redirecting Christina's attention to the features of the activity.

Acting like a mathematician and a computer designer when building games

Christina decided to create a treasure hunting game for six players. The players were playing in turns and their aim was to find the treasure that was hidden behind one of the doors (figure 3). After creating the garden and the outcome card, Christina argued that she would like to play somewhere towards the end because that will increase her chances of winning. She was talking about the chance of getting an outcome by taking for granted that the previous player played and did not get the treasure. However after my intervention she decided to figure out the chance of getting the

treasure for each player before playing the game. In doing so she had to find all the possible combinations and reason why the first and the second players' chances of winning were the same (lines 343-351, extract 3).



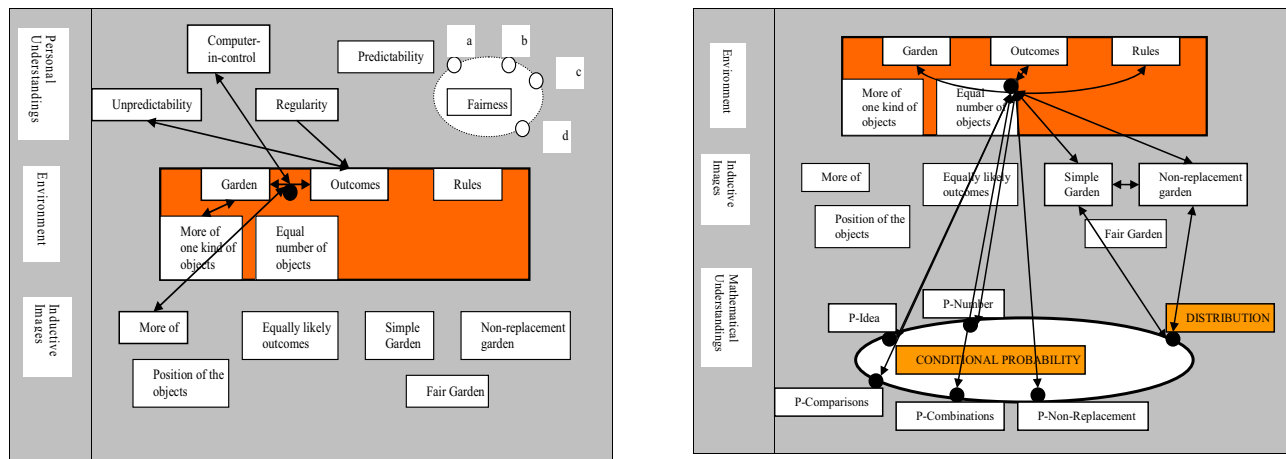
Figure 3: Christina's Game

When creating her game Christina's thinking was mathematical. She was forging connections within the features of the environment and her sense making where one was shaped by the other. In the mediated action taking place during this phase, the role of the agent and the mediational means were redefined. As it is shown in figure 4 (b), Christina was actively drawing on previously constructed connections within the features of the environment (the garden, the frequencies of the objects inside it, the outcomes and the rules) to build her game.

Interview

343. C: **I think that my game is fair...at least for the two first players.** SI
344. R: why do you say that?
345. C: **because they have the same chance of finding the treasure!** M
346. R: so you wouldn't mind if you played first or second?
347. C: **... no, I think that I would like to play first....** SI
348. R: why do you say that?
349. C: **because the first player has 1/6 chance of winning with no conditions whereas the second player has 1/6 chance of winning as well but there is always the chance of the first player winning and that means that the second player will not get the chance to play... So I would prefer to play first than second.** M
350. R: what about if you play first and you don't win, what is the chance of the second player to win?
351. C: **then the second player will have 1 out of 5 chances of winning. It is a bigger chance than the one the first player had but the second player has this chance only if I play and one of the doors is gone from the garden without any winners.** M

Extract 3: Christina's task interview, her game



a) The network of understandings during the 'Lucky Fishing' game b) The network of understandings when Christina was creating her game

Figure 4: Christina's understandings when playing and creating games

Discussion

The children came to this project with a variety of initial understandings which were constructed throughout their in-school and out-of-school experiences with random phenomena. As a result, the children used different personal understandings in their primary articulations. Nevertheless, in each case this variation in initial understandings had the opportunity to be explored within the mediational means available, and through the mediated action which took place new understandings were produced and developed. Although each game created in the third phase was different, and although children's initial understandings were different, through the mediated action taking place in this project, where the specific mediational mean was used, the evolution in children's probabilistic thinking was analogous.

This analogous evolution in children's probabilistic thinking could be explained if we draw attention to Wilensky's (1993), Noss and Hoyles' (1996) and Pratt's (2000) ideas; as well as to the notion of mediated action as used by Wertsch (1998). Wertsch (1998) suggested focusing on mediated action that is the irreducible tension between agents and cultural tools, because in their interaction the boundaries between them begin to erode and the very notion of agent is redefined. Previous to this project, children's informal experiences with probability were formed while interacting with the world and therefore the mediated action taking place was restricted to making sense of chance situations such as lotteries and games with dice and coins, in which the structure of the games and the mathematics embedded in them were hidden from the children. Similarly, children's experiences with formal probability came about from their early in-school encounters, in which understandings formed in their everyday experiences were ignored. In both instances, children's role in mediated action could be identified as an outsider trying to make sense of means that are not accessible to him because they are set out by others; means that have no other

purpose than trying to guess the next outcome correctly; and have no usefulness in other contexts than the ones they are presented in.

In his research Wilensky (1993) defines concreteness as ‘the property which measures the degree of our relatedness to the object (the richness of our representations, interactions, connections with the object), how close we are to it, or as the quality of our relationship with the object’ (p.58). This understanding of concreteness points out to another important element of this analysis, the setting, and in terms of Wertsch’s (1998) analysis, the role of mediational means. Since concreteness is achieved through connections we make with objects we interact with, the role of appropriate and well thought out settings is fundamental. Noss and Hoyles (1996), Pratt (2000), Pratt and Noss (2002), indicate the dialectical relationship between internal and external resources and argue that one shapes the structure of the other.

In the third phase of the project Christina used mathematical understandings and showed an awareness of the properties of the probability tools constructed within her game, which suggests that in Wilensky’s (1993) terms, she had concretized the concept; she had enough modes and models of interaction with it (through playing and building games in which, in the latter, she could not only observe the behaviour of the tools but act upon them with the purpose of manipulating luck and fairness) and therefore she could apply it while constructing her game and use it in her thinking. However, this complex system of understandings was made possible because a specific environment was used in which Christina’s personal understandings were tested and reshaped and new understandings were constructed while Christina was forging connections between her sense making and the exposable tools created within the microworld.

Educational implications

The study reported in this article gives support to the potential for success in the implementation of probability in primary schools, and also suggests what may need to be done for that potential to be realised. The study indicates the importance of using meaningful and suitable settings and tasks for probability in which children’s understandings can be exposed, explored and reformed while at the same time new understandings which draw on the characteristics of formal probability can be produced and developed. In addition, since children have different previous experiences with probability which they bring to any task, the testing of and reflecting on personal understandings during activity in these helpful contexts has to come from the learner in order to be meaningful and to acquire priority and find connections to other concepts. This is best achieved in an environment designed to provoke thinking by providing connections between the tools and the children’s sense making. A computer-based microworld in which children can not only observe the behaviour of the tools but also go beneath the surface in order to manipulate and reconstruct tools while simultaneously constructing their own learning is particularly beneficial. The study has shown that such tools can force reflection in probability

learning. It is also reasonable to suggest that similar means would promote meaningful learning in other aspects of mathematics - but further research is needed to establish which ones.

References

Carraher, T.N., Carraher, D.W. & Schliemann, A.D. (1985). Street mathematics and school mathematics. *British Journal of Developmental Psychology*, 3, 21-29.

Fischbein, E. & Gazit, A. (1984). Does the teaching of probability improve probabilistic intuitions? An exploratory study. *Educational Studies in Mathematics*, 15, 1-24.

Goldstein, R., Noss, R., Kalas, I. & Pratt, D., (2001). *Building rules*, in M. Beynon, C.L. Nehaniv & K.Deutenhahn (Eds), *Proceedings of the 4th International Conference of Cognitive Technology CT2001*, 267- 281. University of Warwick. Coventry: UK.

Goldstein, R. & Pratt, D., (2001). Michael's computer game: A case of open modelling, in M. van der Heuval- Panhuizen (Ed), *Proceedings of the Twenty fifth Annual Conference of the International Group for the Psychology of Mathematics*, 3, 49-56. Utrecht: The Netherlands.

Harel, I. & Papert, S., (1991). Software Design as a Learning Environment. Internet address: <http://lcs.www.media.mit.edu/groups/el/elmemo/7-90memo/>

Hawkins., A.S. & Kapadia., R., (1984). Children's conceptions of probability. A psychological and pedagogical review. *Educational studies in Mathematics*, 15, 349-377.

Hoyles, C., & Healy, L., (1997). Unfolding meanings for reflective symmetry. *International Journal of Computers for Mathematical Learning*, 2, 27-59.

Kahn, K., (1999). A computer game to teach programming. *Proceedings of the National Educational Computing Conference June 1999*. Internet address: <http://www.toontalk.com/english/papers.htm>

Kahneman, D. & Tversky, A., (1972). On the psychology of prediction. *Psychological Review*, 80 (4), 237-251.

Konold, C., (1989). Informal conceptions of probability. *Cognition and Instruction*, 6, 59- 98.

Lave, J. (1988). *Cognition in Practice*. Cambridge, Cambridge University Press.

LeCoutre, M.P., (1992). Cognitive models and problem spaces in 'purely random' situations. *Educational Studies in Mathematics*, 23, 557-568.

Noss, R., Healy, L., & Hoyles, C., (1997). The construction of mathematical meanings: connecting the visual with the symbolic. *Educational Studies In Mathematics*. 33, 203-233.

- Noss, R., & Hoyles, C., (1996). The visibility of meanings: modelling the mathematics of banking. *International Journal of Computers for Mathematical Learning*, 1 (17), 3-31.
- Papert, S., (1980). *Mindstorms: Children, Computers and Powerful ideas*. New York, Basic Books.
- Piaget, J. & Inhelder, B., (1975). *The Origin of the Idea of Chance in Children*. New York, Norton.
- Pratt, D., (1998). The co-ordination of meanings for randomness. *Journal for the Learning of Mathematics*, 18 (3), 2-11.
- Pratt, D., (2000). Making sense of the total of two dice. *Journal for Research in Mathematics Education*, 31 (5), 606-625.
- Pratt, D., Jones I., & Prodromu, T. (2006) An elaboration of the design construct of phenomenalisation, *7th International Conference on Teaching Statistics, Brazil*.
Internet address:
www2.warwick.ac.uk/cll/skills/eportfolio/students/eportfoliodirectory/current/edrfae/publications
- Pratt, D., & Noss, R., (2002). The microevolution of mathematical knowledge: The case of randomness. *Journal of the Learning Sciences*, 11 (4), 453-488.
- Wertsch, J.V, (1998). *Mind as Action*. Oxford, Oxford University Press.
- Wilensky, U., (1993). *Connected Mathematics - Building Concrete Relationships with Mathematical Knowledge*. Ph.D. Thesis, Massachusetts Institute of Technology.

EDUCATIONAL SOFTWARE BASED ON THE THEORY OF CONSTRUCTIVISM

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Abstract: This paper deals with using of educational software on mathematics lessons at 2nd grade of primary school or lower secondary school (pupils age 11-15). We compare educational software based on the Theory of Constructivism and software used only for practice of taught knowledge and skills. We describe developing our software for teaching/learning of the Integers, definition of Constructivism and “good” software.

INTRODUCTION

There is a lot of educational software which can be used in the mathematics educational process. There are many story books, software for practice (e.g. solve these equations...), software for demonstrating, modelling and so on. Most of this software is focused on high schools or universities (Mathematica, Derive, Equation Grapher,...). There exists “good” software usable at lower secondary school, but only if we want to teach geometry – Cabri geometrie. But what we can do if we want to teach arithmetic or algebra by computers at the lower secondary school? Of course, we can use one of the software like Algebra One-To-One to practice some knowledge, or one of products by Fun Math and so on. There is other question: Is it enough? Is it right to use computer only for practice existing knowledge? We think that no. We would like to use software during whole time which is necessary for the teaching of some parts of mathematics.

BETTER SOFTWARE FOR OUR PUPILS

If I would ask you for describing very good software for teaching algebra at lower secondary school, do you answer me? You can do this for yourself. I try to share my visions and ideas about good software for this part of mathematics. At first I have to tell you, what we teach in 2nd grade of the primary school in the Slovak Republic (from algebra and arithmetic) – there are some very difficult parts for pupils in our school. It's Fractions, Percentages and Negative (signed) Integers. They also learn squaring polynomials using formula $(A \pm B)^2 = A^2 \pm 2AB + B^2$, simplification of terms and linear equations. In our research we focused on the most problematic part – negative integers.

What will we be understand as a “good” software? We will mean by “good” software every programs which are in native tongue, it is not necessary to learn syntax and command, programs, where pupils can construct their own knowledge.

We tried to find on the Internet some software usable in this part of mathematics. We found a lot of such software, but it was almost always not the “good” software. There was some of criteria broken – software was only for practising, on have no localization, do not support to constructing knowledge and so on. We don’t think that verifying results it’s the best way how to teach pupils mathematics. Our idea about good software is based on the Theory of Constructivism. We think that software based on this theory can help to understand many of the mathematical terms, operations, causalities and ideas. This software also has to help the pupils construct their knowledge not only from the mathematical point of view but also from point of view of key competences.

THEORY OF CONSTRUCTIVISM AND EDUCATIONAL SOFTWARE

According to Constructivism people construct their own understanding and knowledge of the world through experiencing things and reflecting on those experience. When learners encounter something new they reconcile it with previous knowledge and experience. They may change what they believe in or they may discard the new information as irrelevant. Constructivism modifies the role of the teacher so that teachers help students to construct knowledge rather than reproduce a series of facts. The father of constructivism is a Swiss psychologist J. Piaget. Piaget work has identified four major stages of cognitive growth that emerge from birth to about the age of 14-16. There are:

- Sensorimotor stage (birth to 2 years) – Infants use sensory and motor capabilities to explore and gain understanding of their environments.
- Preoperational stage (age 2 to 7 years) – Children begin to use symbols. They respond to objects and events according to how they appear to be
- Concrete operational (age 7 to 11 years) – Children begin to think logically
- Formal operation (from 11 years) – Children begin to think about thinking. Thoughts are systematic and abstract

The learner is progressing through three mechanisms:

- Assimilation - fitting a new experience into an existing mental structure (schema).
- Accomodation - revising an existing schema because of new experience.
- Equilibrium - seeking cognitive stability through assimilation and accomodation.

There are many historical figures that influence the Constructivism – Jean Piaget, John Dewey, Ernst von Glasersfeld, Lev Vygotsky, Jerome Bruner and others. “The psychological Theory of Constructivism came from Jean Piaget and Lev Vygotsky. Widespread interests of this theory have led to a debate between those who place more emphasis on the individual cognitive structuring process and those who emphasize the social effects on learning” (Fosnot, 1996). The terms “cognitive

constructivism” and “social constructivism” have become common when talking about this psychological theory. Social Constructivism and Cognitive Constructivism are not only types of constructivism. There exist as many theories as many scientist. Social and Cognitive Constructivism be more popular than the others.

All of these psychologists, who had major influences on the theory of constructivism, felt that the teacher is a very vital part of the theory. A constructivist teacher sets up problems and monitors students’ exploration, guides the direction of student inquiry and promotes new patterns of thinking (Brooks & Brooks, 1995)

The constructivist teacher provides tools such as problem-solving and inquiry-based learning activities so that students can formulate and test their ideas, draw conclusions and inferences, and convey their knowledge in a collaborative learning environment.

There are ten general principles of learning that are derived from constructivism. These nine principles are:

- (1) **It takes time to learn:** learning is not instantaneous. For significant learning we need to revisit ideas, ponder them, try them out, play with them and use them. This cannot happen in 5-10 minutes.
- (2) **Learning is an active process in which the learner uses sensory input and constructs meaning out of it:** Learners need to do something, because learning involves the learners engaging with the world.
- (3) **People learn to learn as they learn:** learning consists both of constructing meaning and constructing systems of meaning, each meaning we construct makes us better able to give meaning to other sensations which can fit similar pattern.
- (4) **The crucial action of constructing meaning is mental:** it happens in the mind. Physical actions and hands on experience may be necessary for learning, especially for children, but is not sufficient. We need to provide activities which engage the mind as well as the hands. Dewey called this reflective activity.
- (5) **Learning involves language:** the language we use influences learning. People talk to themselves as they learn, and language and learning are inextricably intertwined.
- (6) **Learning is a social activity:** our learning is intimately associated with our connection with other human beings, our teachers, our peers, and our family. Conversations, interaction with others and collaborations are an integral aspect of learning.
- (7) **Learning is contextual:** we do not learn isolated facts and theories in some abstract ethereal land of the mind separate from rest of our lives. We learn in relationship to what else we know, what we believe, our prejudices and our fears.

- (8) **One needs knowledge to learn:** it is not possible to assimilate new knowledge without having some structure developed from previous knowledge to build on. The more we know the more we can learn.
- (9) **Learning is not the passive acceptance of knowledge which exists "out there".** Learning involves the learner engaging with the world and extracting meaning from his/her experiences.
- (10) **Motivation is a key component in learning.** Not only is the case that motivation helps learning, it is essential for learning.

Our target in research was to find the way how to develop software based on this theory. It was meant to find good environment for modelling and working with mathematical objects which can help to the teacher to teach in constructivist way. We suggested and developed software for chosen themes from algebra. In this article we will focus on the one of themes mentioned sooner – Integers.

In the research we choice 3 classes on the 2nd grade of primary school (age 11-12) by the using of methods for the choice of experimental class (by Turek, 1998) we choice 6A as the 1st group (no software), 6B as the 2nd group (using software only for practicing) and 6C as the 3rd group (using our software).

We used observation and pedagogical experiment to verify our hypothesis:

H1: Pupils which used educational software based on Theory of Constructivism in educational process will take at least equal results as pupils, which used other type of educational software or educational software didn't use.

H2: Pupils which used educational software based on Theory of Constructivism in educational process will have more durable knowledge from taught theme than pupils, which didn't use this type of software.

We developed simple software for teaching signed integers. We used criteria for developing of educational software mentioned in the work of Kalaš (2004 a). In his work he wrote that good educational software is supposed to have two parts – part for explanation and part for practicing, there is also some “story” or “fairy tale”, pictures, sounds, good background and so on. We tried to develop our software as well as possible using these criteria.

DEVELOPPING OF THE SOFTWARE

As we mentioned before we tried to suggest and develop software based on the Theory of Constructivism. First we started with finding environment. There were a lot of ideas about this

- 2D game: going right or left depend on the sign before the number, minus mean left, plus mean right. There were also more ideas – going into the labyrinth, going from point to point, walking on the axes, and so on.
- Mole: the mole is making tunnels, going up and down. The direction was given by the sign.

- Cards: we have red and blue cards, red one means plus, blue one means minus, we must compute with them.
- Store: we have store with packs of something, we can receive some packs (plus) or we can sell some packs (minus).

These activities were only for addition and subtraction. Some of them are didactical games and their use is describe in the Slavíčková, (2006 a). We decide for the last one after interview with pupils of 11-12 ages. The activities for multiplication and division are inspired by the history.

For multiplication of negative and positive integers we use again “real life” environment. There are some animals that live in the water. They would like to build a house but they need to borrow money. How much money they must borrow if they need X (pupils can change this number) packs of the bricks and they know how much the prize for one pack of the bricks? It is similar for division; we use the model of the ant-hill and division of debts between the bugs living in.

For multiplication with negative numbers we use the sequence of the product of the positive and negative number, for instance:

$$4.(-3) = -12$$

$$3.(-3) = -9$$

$$2.(-3) = -6$$

$$1.(-3) = -3$$

$$0.(-3) = 0$$

$$(-1).(-3) = ?$$

We can see the relation between the first member of product and the result if the second member of the product is the same. This we can use in practise. Similar model we use for division of two negative numbers.

For better understanding of how the software works we shortly describe two lessons where was this software used:

First lesson in the 3rd group

After the pupils took seats we told them to run program ZC.htm and then to choose part “LEARNING” and then “ADDITION AND SUBTRACTION”. The pupils saw next picture:

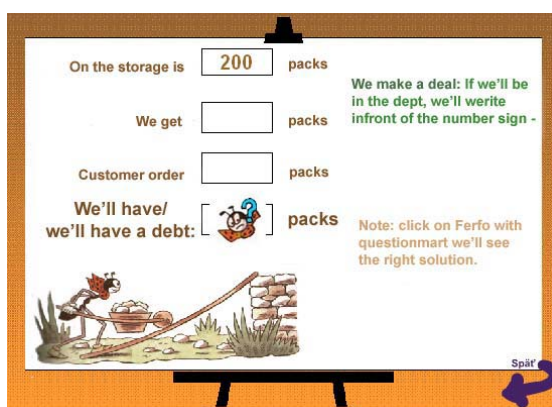


Figure 1: screenshot of the part for learning addition and subtraction

We asked them to “play” with this part (5 min) and to write their reflections about computing into their sheets. After 5 minutes we asked them for their notes. We didn’t comment this notes, just listen and then we started to give them a lot of question and tasks:

Do you know how to take 100 packs on the storage? Do it!

After this everybody had 100 packs on the storage and we could ask them for doing some operations with it:

T1: First customers are coming. They’d like to buy 50 packs. Can you solve it to them? Do it!

T2: There is new delivery from central storage and you receive 200 packs.

We gave to pupil tasks like T1 and T2. After everybody knew what to do, we asked them for more packs than is on the storage. For instance, we asked for 250 packs but on storage were only 200 packs. Then we asked them:

Question 1: How many packs we have now?

Question 2: What we can do to have no dept?

Question 3: What we must to do, to have at least 1 pack on the storage?

Afterwards we performed a lot of ordering and delivering, but we tried to stay in the negative integers as long as possible. We could simulate these kinds of computation: $-a-b$, $-a+b$, $a-b$, $a+b$. The most important for us were the first and the second one. We didn’t tell the pupils about it.

At the end of lesson we asked them for writing mathematical notation of these activities into their sheets. We started with this notation on the next lesson.

Second lesson in the 3rd group

We started with retrieving last lesson and we asked the pupils for their notation. We found the correct notation for it and we started to use it. Then we solved exercises from school-book or from worksheet. We focused on exercises like this $50-80=$, $10-25=$ and so on. The methodology of solving these exercises:

We have 50 packs on storage. The customer asks us for 80 packs. How many packs we’ll have on the storage?

We solved exercises like this about 30 minutes. The last 15 minutes we asked the pupils for solving these problems:

Problem 1: What happened, if we find exercise like $-80+50=?$ How we can interpret this? How we can solve it?

Problem 2: What happened, if we find exercise like $-80 - 50 = ?$ How we can interpret this and how we can solve it?

Problem 3: What is different between result exercise $-80 + 50 =$ and exercise $50 - 80 = ?$ How we interpret it? How we solve it?

When the pupils solve these problems, we made from this some general results and they wrote them into their sheets. After this we found other problem – in the school-books are exercises like $-80 + (-50) =$. We formulated this like other problem and asked pupils for the solution on the next lesson.

THE EXPERIMENT

In the 1st and the 2nd group we taught “classically”, it meant – we stood in front of blackboard and write some definition, examples and then we asked pupils for their solution of exercises in the school book. In the 3rd group we used our software on the lessons. Description 2 of 17 lessons which we taught in the school year 2004/2005 in the primary school in the Bratislava is before.

We gave the pupils a test at the end of the experiment to verify our hypotheses. For evaluating our experiment we used statistical evaluating of the hypotheses (F-test) and also “atomic analysis” (Hejný - Michalcová, 1999). Atomic analysis is based on the dividing the solution into the smallest steps – atoms. Teacher (experimentator) tries to find the reason of these steps. We do not take an attention to this method in this paper.

We find some very interesting solution of the 5th exercise:

Exercise: Ondro owes to Janko 70Sk. Janko owes to Peter 90Sk and Peter owes to Ondro 40Sk. What will be the difference between old and current sum of their money?

Reactions and solutions of pupils in the 1st group (no software on lessons)

Pupil 1: How can Janko get to the Peter 90Sk if he has only 70 crowns?

Pupil 2: $90 - 70 = 20$, $70 - 40 = 30$

Reactions and solutions of pupils in the 2nd group (use educational software only for practicing)

Pupil 1: $70 - 40 = 30$, $90 - 30 = 60$. Difference is 60.

Pupil 2: $70 + 90 + 40 = 200$, $200 - 70 = 130$,
 $200 - 90 = 110$, $200 - 40 = 160$.

Using the atomic analysis we try to find answer for the question: “Why the pupils in the 1st and the 2nd group solve this exercise so bad?” We think, that is because of few “touch” with exercises from “real life”. In the school book are a lot of exercises without the context, simple “Solve!”

Pupils in the 3rd group solve this exercise without big mistakes. Their successfulness was 78%. Successfulness in the 1st group was only 50% and in the 2nd group 59%. We can not say much about hypothesis H2 because less statistical significance. It will need deeper and wider experiment – more pupils, more schools, different parts of Slovakia or other country, etc.

For comparing all three groups by a statistical way we used an ANOVA test based on Fisher test. The value for F-test criterion was 0,358 what's the probability of difference 65%. So we can say, that there's no statistical difference between these three groups. Hypothesis H1 is valid.

CONCLUSION

In this paper we described two lessons from our experiment. This one was dealing with two hypotheses about teaching and learning by our educational software and verification of these hypotheses.

Some parts of described software are didactical games so we can use these didactical games also without computer (see experiments in Vankúš, 2005). We can see that educational software can help our pupils to understand better tasks from real life (sometimes everyone must owe some money) and thus prepare pupils for the main goal of the lower secondary school - the preparation of children for their future life. So using software based on the Theory of Constructivism is a good choice for every teacher.

Unfortunately there is not enough “good” software, so we must cooperate with programmers and tell them about our ideas and activities for teaching by computer. We mention about simple software for teaching Integers. Developing of this software was not easy and we do not think that it is already finished. Pupils are year by year different and more comfortable with computer so it is not bad idea to make more programs for them. It is better to have more experiences with different models of integers, fraction and so on. The pupils then can better make a net of knowledge and construct new knowledge.

We describe only one type of software – based on the Theory of Constructivism. It is because we think that is better for using in the school than the other one. Pupils can the story books, hypertext and software for practising knowledge as well use at home for improvement their knowledge.

BIBLIOGRAPHY

- BROOKS, J.G., BROOKS, M.G. (1995). *Constructing Knowledge in the Classroom*. February 2006 for Internet. <http://www.sedl.org/scimath/compass/v01n03/1.html>.
- DEWEY, J. (1916): *Democracy and Education. An introduction to the philosophy of education* (1966 edn.), New York: Free Press.

- FOSNOT C.T. (1996). *Constructivism: Theory, Perspectives, and Practice*. New York: Teachers College Press.
- HEJNÝ M., NÔTA S.(1990): Methodology of the negative numbers on elementary school, *Matematické obzory*, Zväzok 35/1990, Alfa Bratislava, 1990
- HEJNÝ M., MICHALCOVÁ A.(1999): Analysis of the student's solving in mathematics test, *Metodické centrum Banská Bystrica*, 1999
- HEJNÝ, M.(2004): Negative numbers, In: (Ed). M. Hejný, J. Novotná, N. Stehlíková: *Dvacet pět kapitol z didaktiky matematiky 1*, Univerzita Karlova v Praze - Pedagogická fakulta, Praha, 2004, ISBN 80-7290-189-3 (1. sv.) p. 327-342
- KALAŠ I.(2004 a): Developing of the educational software, www.edi.fmph.uniba.sk/vyuka, June 2004
- KALAŠ I.(2004 b): Using ICT in educational process, www.infovek.sk/konferencie, June 2004
- PETLÁK E.(2000): Pedagogical and didactical work of the teacher, IRIS Bratislava 2000
- PIAGET J., INHELDER B.(1997): *Psychológia dieťaťa*, SOFA 1997
- PUGA A. B.(2002): About the success of integration of information technology in math education, *Proceeding of the International Conference: The Humanistic Renaissance in Mathematics Education*, Palermo 2002 (<http://math.unipa.it/~grim/palermo2002.htm>)
- SLAVÍČKOVÁ M.(2006 a): Rôznorodé metódy vyučovania tematického celku Záporné čísla, *Zborník príspevkov z konferencie v Ružomberku*, september 2005
- SLAVÍČKOVÁ M.(2006 b): Software on the mathematical lessons on primary school, *Transformácia vzdelávania smerom k potrebám európskeho trhu práce*, Fórum pedagogiky, Bratislava 2006, ISBN 80-8052-255-3, p. 288-295
- TUREK I.(1998): *Teacher and research in didactics*, MC Bratislava, 1998
- VANKÚŠ P.(2005): Efficacy of teaching mathematics with method of didactical games in a–didactic situation. In: *Quaderni di Ricerca in Didattica*, No. 15, G.R.I.M, University of Palermo 2005, ISSN 1592-5137, s. 90–105
- VERMETTE, P.; FOOTE, C.; BIRD, C.; MESIBOV, D.; HARRIS-EWING, S.; & BATTAGLIA, C. (2001): Understanding Constructivism: A Primer for Parents and School Board Members. *Education*, 122, (1), 87-93.
- F-UNDERSTANDING: <http://www.funderstanding.com/constructivism.cfm>, (Feb. 2006)

TEACHING WITH A SYMBOLIC CALCULATOR – THE FIRST YEARS OF A LONG-TERM PROJECT

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A six-year project was started in autumn 2003 to test the use of scientific calculators in grammar schools in Bavaria (Germany). The aim of the project was to answer the following questions:

- *How have basic mathematical skills (e.g. algebraic transformations, solving equations, working with tables and formulas) changed?*
- *How have the questions posed in examinations changed when the students are allowed to use a scientific calculator (with CAS)?*
- *How have the students evaluated the use of the new tool?*
- *How have teaching styles and methods changed in mathematics lessons?*

The project should prove the practicability of using symbolic calculators in regular maths lessons. This article presents the results of the first three years of the project.

THEORETICAL BACKGROUND

Even though students in many countries are allowed to use symbolic calculators (with CAS) for schoolwork and in examinations, they have only been partially integrated into mathematics teaching worldwide (in contrast to graphics calculators, which have become mandatory in the majority of countries). There is a variety of reasons for this: they involve the tool itself (i.e. complexity of use, low resolution of the graphics screen, lack of a pedagogical tool), the views of the teachers (i.e. insufficient familiarity with the tool, concern about low abilities of the students, importance of traditional mathematics) and also the curricula (i.e. inadequate integration of the new tool into the goals of mathematics lessons). Moreover, the complex area of integrating the technology has been underestimated (see Trouche 2005).

In recent times the “theory of instrumental genesis” has been developed to describe the process of integrating calculators into mathematics education. The focus of this theory is to develop learning environments or an “instrumental orchestration”, in which the calculator (the artefact), or tool, changes into an *instrument* that’s useful for solving problems and that mediates between the user and the mathematical contents (see Drijvers & Herwaarden 2002, Artigue 2002, Drijvers & Gravemeijer 2005).

In the past years, many empirical investigations concerning the use of CAS or scientific calculators (with CAS) in the teaching of mathematics have been published (see Lagrange 1999, Peschek & Schneider 2002, Weigand & Weller 2001, Guin et.

al. 2005). These studies show that the use of a CAS brings a greater meaning to work with diagrams, reinforces experimental work in which the assumptions were obtained through systematic testing and that CAS appears to bring an increase in automatic work as well as computer cooperative forms of work. However, many investigations only looked into the use of the computer for a few weeks (Schneider 2000, Barzel & Möller 2001, Drijvers 2003, Pierce & Stacey 2004), and thus do not show the long-term effects on the knowledge and ability of the students.

In the following, the evaluation of the first three years of the project on the use of SC in the 10th school year of Bavarian grammar schools is presented.

THE TEACHING PROJECT

In the school year 2003/04, a total of 137 students from six Year 10 classes (referred to in the following as SC-classes) in three Bavarian grammar schools used the TI Voyage 200 (referred to here as a symbolic calculator or SC) in lessons and exams for one year. Four Year 10 classes (121 students in total) were included as control classes. The project was repeated in the school year 2004/05 with ten Year 10 classes (118 students used the SC and 126 students were in the control classes) and again in the school year 2005/06 with twenty-two Year 10 classes (257 SC and 145 in the control). The project will continue in subsequent years in Years 11, 12 and 13.

The following topics were taught: calculating with powers and power rules, power functions, the exponential and logarithmic functions, measurements of circles, trigonometry, the volume and surface area of cylinders, cones and spheres.

The study was intended to answer the following questions:

1. What changes in core mathematical abilities of the students in SC-classes could be identified after one year? (E.g. formation of terms, interpretation of graphs, solving equations, working with tables and formulas.)
2. Does – as is frequently claimed – the difference between ‘good’ and ‘poor’ students increase with the use of the SC?
3. What kinds of attitudes or beliefs develop among students using the SC? Are there differences between male and female students?
4. How do examination questions change as a result of the use of the SC?
5. How are the capabilities of the students developed in learning to use the SC effectively?
6. How do the teachers evaluate their teaching lessons and their change through the use of the SC themselves?

TEST APPARATUS

Questions 1 and 2 were answered by a pre- and post-test design.¹ All tests were taken using paper and pencil, the use of SC was not allowed. The questions were about basic mathematical skills like the manipulation of terms but there were also questions involving interpretation of graphs and transfers between different representations. To answer the third question, a questionnaire was developed with answers given according to a five-point scale and questions allowing open answers. For the fourth question, external experts evaluated the regularly written exam questions of the SC-classes with respect to the expected competencies of the students and possible solving strategies (while using a SC).

Research question four should give hints as to whether the teachers integrated more questions – compared to “traditional” examinations – concerning process-oriented competencies like problem solving or modelling.² To answer the fifth question, the SC-classes had to pass a post-test with SC as well.³ This test was a new feature in school year 2005/06 and was not given the years before. For the sixth question, the teachers of the SC-classes kept a log of teaching hours and recorded themes of lessons, teaching time using the SC and the method of teaching while using the SC.

In the following, the results of the test are presented along the lines of the six research questions.

RESULTS

Student Achievements (Question 1)

At the start of the school years the SC-classes and control classes took a pre-test. At the end of the school years the post-tests were taken. The diagrams in figs. 1, 2 and 3 compare the results from the post-test for the SC and the control groups.

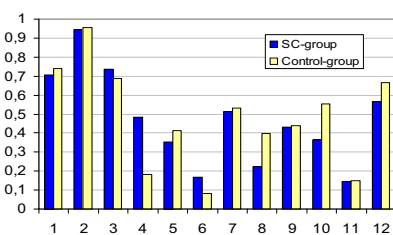


Fig. 1: School year 2003/04.

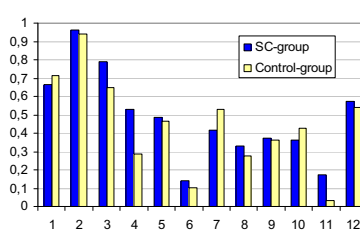


Fig. 2: Year 2004/05.

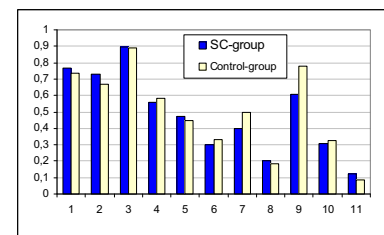


Fig. 3: Year 2005/06.⁴

Test results for the SC and control groups

The results show that the students in the SC-classes achieved better results when working with graphs of functions and doing transfers between equations of

¹ You will find the test questions on www.didaktik.mathematik.uni-wuerzburg.de/weigand/2007/cerme5

² To answer the question whether or how process-oriented competencies have been integrated into the lessons, the lessons had to be (video-taped and) evaluated by a researcher. This will be done in the up-coming phase of the project.

³ See also: www.didaktik.mathematik.uni-wuerzburg.de/weigand/2007/cerme5

⁴ The test questions only changed slightly over the years. Question 12 was cancelled in school year 2005/06.

functions.⁵ No differences between the SC and control classes were observed when working with variables, terms and tables. This discredits the recurring argument that algebraic skills stay underdeveloped with the use of computers. The results did however show a decline in the ability of students in the SC-classes to solve equations of the type $x^2 + 5x = 0$ or $\sin x = 0.5$. The reasons for this are not obvious from the present data.

CAS BENEFITS AND STUDENT DIFFERENCES (QUESTION 2)

In addition, the pre-test results were also used to separate the poorly performing and highly performing groups of students. Based on the total number of points scored on the pre-test, the students were divided into three groups: poor, average and good.⁶ The performance of each of these groups was assessed. (Fig. 4 and Fig. 5).

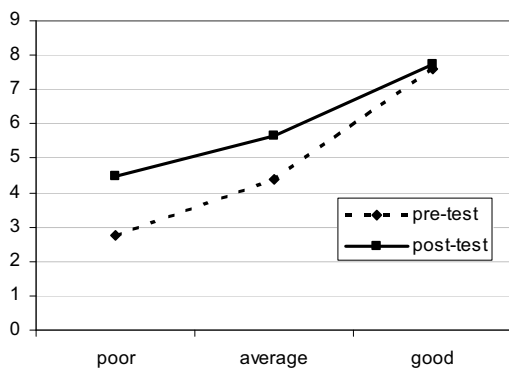


Fig 4: Results of the SC-group for the school year 2004/05.⁷

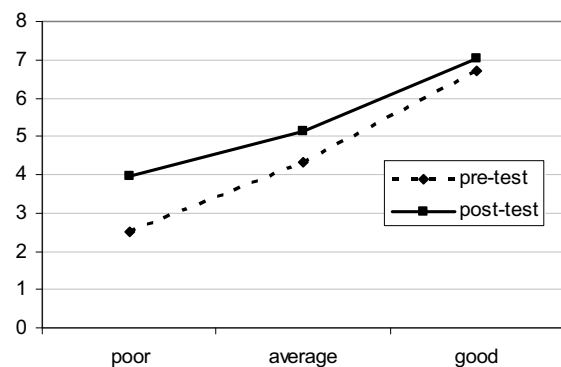


Fig. 5: Results of the control group for the school year 2004/05.

One can see that the “scissors effect”, namely that good students get better and poor students become even poorer, did not occur here. On the contrary, there was an improved performance particularly among the poor and average groups, whilst the ‘good’ students improved only slightly.

There are only hypotheses for explaining these results. Maybe the “good” students were not motivated by the test questions, which presupposed only absolutely necessary basic knowledge. There are a few hints in this direction.

The results also clearly show differences between boys and girls. Standard problems like term manipulations were solved better by boys, whereas “creative problems” like finding formulas from tables were better solved by girls. It is also surprising that the achievement of girls in the control-group increased much more than that of girls in

⁵ The tests were statistically compared with the t-test.

⁶ The average score in the pre-trial test was around 4.5 points. The groups were classified according to the following criteria: poor performing: less than 3.5 points (34 students), average: between 3.5 and 5.5 points (61 students): high performing: more than 5.5 points (30 students).

⁷ We have got nearly the same results for the other school years.

the SC-group. This might be due to the fact that girls may have a negative attitude to the SC and were therefore not sufficiently motivated.

STUDENT BELIEFS AND ATTITUDES (QUESTION 3)

The SC-classes filled out a questionnaire evaluating the lessons with SC. The students were first asked to comment using a five-point scale, and then had to answer three open-ended questions. The following shows the distribution of the answers in percentages:⁸

		++ ⁹	+	O	-	--
1.	The lessons with the Voyage 200 were more interesting than previous lessons.	16 %	40 %	22 %	11 %	10 %
2.	The lessons were easier because of the Voyage 200.	10 %	36 %	22 %	19 %	13 %
3.	The lessons were more varied.	21 %	47 %	14 %	9 %	8 %
4.	I learnt more in these lessons than in other lessons.	2 %	15 %	44 %	17%	22 %
5.	Mathematics in these lessons gave me more pleasure.	8 %	24 %	36 %	15 %	17 %
6.	Using the Voyage 200 has shown me a whole new side to mathematics.	9 %	35 %	22 %	19 %	15 %
7.	I participated more actively than in other lessons.	4 %	11 %	42%	18 %	25 %
8.	I used the Voyage 200 outside of lessons and homework.	8 %	32 %	11 %	25 %	24 %
9.	I'd really like to carry on using the Voyage 200 in mathematics lessons.	32 %	21 %	8 %	11 %	27 %
10.	I would recommend my friends in Year Nine to go into a class that will use the Voyage 200 without hesitation.	14 %	31 %	9 %	24 %	21%
11.	I used the Voyage 200 frequently for my homework.	17 %	42 %	10 %	19 %	11 %
12.	The Voyage 200 is very easy to use.	9 %	30 %	9 %	35 %	16 %
13.	I frequently spent a long time searching for particular commands for the Voyage 200.	15 %	40 %	5 %	32 %	7 %

The answers to questions 8 – 13 clearly divide the group in two.¹⁰ One group liked working with the calculator, used it outside of the lessons, had no great difficulties in

⁸ Values rounded to the nearest whole number. The results are from the school year 2003/04. The results for the school year 2004/05 are quite similar.

⁹ ++: strongly agree; +: agree; o: neither agree nor disagree; -: disagree; -- : strongly disagree.

using it and would like to continue using it. The other group did not enjoy working with the calculator as much as the first group; they worked with the calculator less outside of the course, and had difficulties using the tool.

The mean values for boys and girls differ for all questions, and for almost all the questions these differences are significant. Boys support the use of SC much more than girls - they think they learned more while using the SC and would recommend the use of the SC to their friends.

ASSESSMENT - THE USE OF SC IN CLASS TESTS (QUESTION 4)

Several different research projects have recently looked at the use of CAS in written exams (BROWN 2003). The results largely agree that the structure and type of questions should not be fundamentally changed, and that the CAS students will have a greater variety of solution strategies available to them and will therefore be able to choose their own strategy.

An analysis of the class tests in this project shows that the vast majority of the questions or problems could be presented in the same way in an examination that did not allow the use of a computer. However the use of SC opens up new possibilities for solving problems, since the students have to decide for themselves which method or strategy they will use to solve the given problem. For example, the zeros of a function can be read directly from the graph, obtained by pressing a button (using the “Zero Command”), using a menu command – solve($f(x)=0,x$) – or with the help of a table of values.

COMPETENCES IN SC USE (QUESTION 5)

For bureaucratic reasons, we didn’t have the possibility of evaluating authentic student solutions to problems in class tests. In school year 2005/06 we gave a post-test in the SC-classes that allowed the use of the SC. This test was designed to answer the following questions:

- How do the students use the SC?
- Which strategies (symbolic, graphic, and numerical) do they use while solving problems?

Students’ strategies are very much related to how the problem is *formulated*, rather than the problem itself. For example, solving an equation like $\cos\left(\frac{1}{5} \cdot x\right) = x^3$, or determining an intersection point of the graphs of two functions, e. g. $f(x) = \sin(x) + 1$ and $g(x) = 2^x$, are – for the students – different problems. The first is solved with the “solve-command” on the SC, while the second is solved graphically. This shows that the SC is used quite mechanically. The results of the test using SC also show mistakes that were directly related to the handling of the calculator. The calculator is

¹⁰ A more detailed evaluation of the questionnaire revealed that there is indeed a stable group of students who gave answers on the “minus-side” for all questions 8 – 13 and a group of students who marked their answer on the “plus-side”.

not a self-explanatory tool or instrument, and to handle it properly is something one needs to *learn*.

SELF-EVALUATION OF THE TEACHING LESSONS (QUESTION 6)

Lesson protocols

The teacher of the model classes filled in a paper protocol about the use of the SC during the school year, in which the topics covered, amount of time spent, method of teaching, and the predominantly used SC-window were recorded each lesson in different teaching phases.

Content of lesson, exercises, worksheets, etc.	Lesson form while using the SC	Predominantly used SC-window	Length of time in minutes (approx.)...
	Teacher centred Individual work Working in pairs Group work Project work	Algebra Window Graphics Window Table Window Geometry Window	

Record of lessons

The entries were normally filled in after the lesson by the teacher and therefore give only a rough estimation of the timeframe of the use of the SC.

The analysis of the protocols shows that the SC was used in around half of the mathematics lessons. In these lessons, around half the time was allocated to the use of the SC. That does not mean that the computer was used constantly during this time, but rather that the computer was used in these lessons.¹¹

With regards to the use of various teaching styles, we can see that about 30% of teaching included working in pairs or groups, and likewise 30% individual work or student presentations. On the basis of this data, no valid empirical conclusion is possible;¹² however, considering the records of traditional (i.e. calculator-free) German language mathematics lessons (e.g. the TIMSS-Video-Study 1997), do at least establish the hypothesis that the calculator is a catalyst for teaching methods. This has been consistently claimed for German mathematics lessons, e.g. in the current TIMSS and PISA debates.

The teachers reflected on their own teaching in two written reports, one halfway through the year and one at the end. We can therefore summarise their statements as follows:

¹¹ This can mean a demonstration of the computer on the part of the teacher or of a student or the occasional use of the calculator by individual students or also systematic work with the computer by the whole class.

¹² For this an exact record of the timeframe of the corresponding teaching forms of the experiment and of the control class would be required.

- Extended contents: Equations can now be solved which, up until now, had only been dealt with in special cases, for example polynomial equations of higher degrees, exponential or logarithmic equations. Furthermore there is an access to types of functions that had not been considered previously, such as polynomial functions, or discrete logistic growth functions (which could be defined by recursive equations). See Weigand 2004.
- Emphasising flexible thinking: Using dynamic diagrams offers the possibility of working intensively with parameter-dependent functions.
- SC as a control instrument of calculations: The control skills of the students had to be developed, but this was frequently only achieved by the stronger performing students. In contrast, poorly performing students did not manage to successfully search for and remove a mistake when there was a discrepancy between the control and original calculations.
- Discrete operations or activities: Content such as recursively defined sequences gain a greater importance, as there is now a tool available that allows the effective algorithmic handling of mathematical objects.
- All of the teachers were of the opinion that particularly the poorly performing students were very passive when working with the calculator and did not familiarise themselves to any great extent with the use of the tool. For these students the computer remained a *tool* or *artefact* and did not develop into an *instrument*.
- The teachers considered the preparation for the lessons to be considerably more time-consuming because of the inclusion of a new medium, and considered the lessons more challenging particularly as a result of many different tests associated with the content and technical problems. In order to get to know how the lessons and the teaching styles *really* changed, a more extended investigation would be necessary (see. e. g. Kandal a. Stacey 2002).

CONCLUSION

The year-long project shows that the SC was well integrated into the regular teaching of the Year 10 classes. It confirms the following findings:

1. The results of the pre- and post-tests confirm an increase in capabilities in the areas in which an SC can be used as an advantage. Thus, the students of the SC-classes achieved a greater improvement, compared to the control class, in working with graphs of functions and the transfer between equations. Furthermore, no difference was detected working with variables, terms and tables. This shows in particular that algebra skills did not stay underdeveloped with the SC-classes.
2. The improved performance amongst the poor and average students is noticeable. The quite often claimed “scissors effect” (good students get better and poor

students become even poorer) did not occur. But it is very precarious that the best quarter of the students of the SC-classes (concerning to the result of the pre-test) did not improve (concerning to the result of the post-test). The hypothesis for further investigations is that these students were “under-challenged” by the used methods of lessons and that up-coming lessons will have to support these students individually.

3. The beliefs and attitudes of students with regard to the new tools are ambivalent. One group was very willing to use the SC and would like to continue using it in future years. The other group did not enjoy working with the computer, used it less outside of the course and had difficulties using it. For almost all the questions, there are significant differences between the results for boys and girls. There are hints that some girls may have started out with a negative attitude to the SC and, as a result, were not sufficiently motivated and therefore didn't improve their achievement.
4. The questions given on the class tests are not substantially different from traditional class tests. The review of experts of the examination questions, compared to traditional examinations, shows that new solution strategies (e.g. graphical, numerical solutions, experimental methods) are possible, that the computer can be used as a heuristic tool (especially in drawing the graphs of functions by just pressing a button) and as a control instrument to check results obtained with pencil and paper. But these new possibilities have to be learned.
5. Many students – even after one year in class – weren't able to use the computer effectively in new problem situations. The calculator is not a self-explanatory tool or instrument, and one has to learn how to use it.
6. At the least, the lesson protocols of the teachers establish the hypothesis that the calculator is a catalyst for “new” teaching methods. Individual work, as well as partner and group work in mathematics lessons appeared to be reinforced. But teachers claim that the poorly performing students were especially very passive when working with the calculator. Hence, not only the top group in the class but also the weaker students need individual support.

Due to the positive results obtained in this project, the Bavarian Ministry decided to continue the project. The follow-up project was started in September 2005 with 10 classes using the SC in Year 10 and they will continue using this new tool over the next 4 years until their final examination. This will allow a systematic investigation of some open questions identified in this project, and offers the possibility of evaluating the development of long-term competencies.

REFERENCES

ARTIGUE, M., Learning Mathematics in a CAS Environment: The Genesis of a Reflection about Instrumentation and the Dialectics between technical and

- conceptual Work, *International Journal of Computers for Mathematical Learning* 7 (2002) 245-274
- BARZEL, B. A. MÖLLER, R., About the Use of the TI-92 for an Open Learning Approach to Power Functions – A Teaching Study, *Zentralblatt für Didaktik der Mathematik* (ZDM) 33 (2001), Bd. 1, 1-5,
- BROWN, R., Computer Algebra Systems and Mathematics Examinations: A comparative Study, *International Journal of Computer Algebra in Mathematics Education* 10 (2003), No. 3, 155-182
- DRIJVERS, P. A. GRAVEMEIJER, K., Computer algebra as an instrument: examples of algebraic schemes, in: Guin, D., Ruthven, K., Trouche, L. (Eds.), *The Didactical Challenge of Symbolic Calculators*, Springer, New York 2005, 163-196
- DRIJVERS, P. A. VAN HERWAARDEN, O., Instrumentation of ICT-tools: The case of algebra in a computer algebra environment, *International Journal of Computer Algebra in Mathematics Education* 7 (2000), 255-275
- GUIN, D., RUTHVEN, K., TROUCHE, L. (Eds.), *The Didactical Challenge of Symbolic Calculators*, Springer, New York 2005
- PESCHEK, W. A. SCHNEIDER, E., Computer algebra systems (CAS) and mathematical communication, *The International Journal of Computer Algebra in Mathematics Education* 9 (2002), 229-242
- PIERCE, R. A. STACEY, K., A Framework for Monitoring Progress and Planning Teaching towards the effective Use of Computer Algebra Systems, *International Journal of Computers for Mathematical Learning* 9 (2004), 59-93
- SCHNEIDER, E., Teacher Experiences with the Use of a CAS in a Mathematics Classroom, *International Journal of Computer Algebra in Mathematics Education* (2000) v. 7(2) 119-141
- TROUCHE, L., Calculators in Mathematics Education: A rapid Evolution of Tools, with differential Effects, in: Guin, D., Ruthven, K., Trouche, L. (Eds.), *The Didactical Challenge of Symbolic Calculators*, Springer, New York 2005, 9-39
- WEIGAND, H.-G., Sequences - Basic Elements for Discrete Mathematics, *Zentralblatt für Didaktik der Mathematik* 36 (2004), Nr. 3, 91-97
- WEIGAND, H.-G. a. WELLER, H., Changes of Working Styles in a Computer Algebra Environment - The Case of Functions, *International Journal of Computers for Mathematical Learning*, Kluwer 6 (2001), 87-111