PLENARIES

CONTENTS

Balancing Interest in Fundamental Understanding with Considerations of Usefulness in Mathematics Education Research[1]

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Overview

I wish to begin by thanking the conference organizers for inviting me to give this lecture. Unfortunately, for various reasons, I have not been able to participate in international conferences and meetings for the past several years. I say unfortunate, not because I have anything special to contribute to these meetings, but for a more selfish reason: I have found that I greatly benefit from the opportunity to learn from the tremendously talented individuals who make up our international research community. Moreover, these meetings are great fun!

When Professor Straesser invited me to give a lecture at this conference, he asked that the me to talk about *research methods*; clearly an important topic, but one that I did not feel particularly well-qualified to take on for a plenary address. But the more I thought about it, the more I came to accept that I really do have something worthwhile to say on this topic. I hope you will agree.

The basic argument I will present in this lecture is that the choice of research methods must be made in view of one's philosophical and theoretical perspectives, what the purpose of doing research should be. I will also argue that the purpose of our research should be to make lasting changes in both practice and policy; that is, it should be transformative in nature. Furthermore, I will argue that there often are forces – both political and ideological – at work that influence the methods we use and, consequently, the sorts of questions we seek to answer. Finally, I will argue that some research methods are more likely to lead to the sorts of transformative change that I am calling for. (*Note*. I strongly urge anyone interested in more complete discussions of my arguments to read the references highlighted in **bold** at the end of this paper. I drew heavily upon them in preparing this lecture.)

My comments are organized into five sections:

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[1] Plenary lecture presented at the Conference of European Research in Mathematics Education, Larnaca, Cyprus, 22 February, 2007

1. *Historical context* for mathematics education research (with special attention given to research methods and the influence of politics)
2. *The roles theory and philosophy in research* (with attention given to matching theory and philosophy to purpose)
3. *The purposes of research* (clearly our research methods should match our purposes)
4. *Types of research activities*
5. The promise of *design research methods*

Establishing a Historical Context

The current emphasis in the United States being placed on so-called *scientific research* in education is driven in large part by political forces. Much of the public discussion has begun with an assumption that the purpose of research is to determine "what works," and the discourse has focused largely on matters of research design and data collection methods.[2] One consequence has been a renewal of attention to experimental designs and quantitative methods that had faded from prominence in education research over the past two decades or so.

Today's debate in the United States over research methods calls to mind the controversy that raged 40 years ago surrounding calls to make mathematics education research (hereafter referred to as MER) more "scientific." A concern voiced by many at that time was that MER was not answering "what works" questions precisely because it was so narrowly embedded in a research paradigm that simply was not up appropriate for answering questions of real importance—specifically, the positivist, "experimental" paradigm (Lester & Lambdin, 2003). Writing in 1967 about the need for a journal devoted to research in mathematics education, Joseph Scandura, an active researcher in the U.S. during the 1960s and 70s observed:

> [M]any thoughtful people are critical of the quality of research in mathematics education. They look at tables of statistical data and they say "So what!" They feel that vital questions go unanswered while means, standard deviations, and *t*-tests pile up. (Scandura 1967, p. iii)

A similar sentiment was expressed in the same year by another prominent U.S. researcher, Robert Davis:

> In a society which has modernized agriculture, medicine, industrial production, communication, transportation, and even warfare as ours has done, it is compelling to ask why we have experienced such difficulty in making more satisfactory improvements in education. (Davis, 1967, p. 53)

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[2] I make no claims about the state of mathematics education research in any countries other than the United States. One can only hope that the situation is not as troubling elsewhere.

Davis insisted then that the community of mathematics education researchers needed to abandon its reliance on experimental and quasi-experimental studies for ones situated in a more interpretive perspective. Put another way, the social, and cultural conditions within which our research must take place require that we adopt perspectives and employ approaches that are very different from those used in fields such as medicine, physics, and agriculture. Today, we education researchers find ourselves in the position of having to defend our resistance to being told that the primary characteristics of educational research that is likely to receive financial support from the U.S. Department of Education are "randomized experiments" and "controlled clinical trials" (U.S. Department of Education, 2002).

To a large extent, the argument against the use of experimental methods has focused on the organizational complexity of schools and the failure of experimental methods used in the past to provide useful, valid knowledge (Cook, 2001). However, largely ignored in the discussions of the nature of educational research methods has been consideration of the conceptual, structural foundations upon which these methods have been created. To be more specific, the role of theory and the nature of the philosophical underpinnings of our research have been absent. This is very unfortunate because scholars in other social science disciplines (e.g., anthropology, psychology, sociology) often justify their research investigations on grounds of developing understanding by building or testing theories and models, and almost always they design their research programs around theoretical frameworks of some sort. In addition, researchers in these disciplines pay close attention to the philosophical assumptions upon which their work is based. In contrast, the current infatuation in the U.S. with "what works" studies seems to leave education researchers with less latitude to conduct studies to advance theoretical and model-building goals and they are expected to adopt philosophical perspectives that often run counter to their own. However, the reality is – at least in the United States – that researchers' professional lives are too often dramatically influenced by the availability of funding from state and federal funding agencies.[3] To be sure, educational researchers have a responsibility to look for answers to societal issues and problems; this is the transformative purpose of our work. But, as long as funding agencies engage in telling researchers *how* to conduct their work, the more impoverished our research will be, thereby resulting in a "piling up" (as Scandura put it) of statistical results that tell us little of value. In a later section of this

[3] It is beyond the scope of this paper to elaborate on the relationship in the United States between funding agencies and universities. It suffices to say that university faculty – who after all conduct the majority of our educational research – are under increasing pressure from their institutions to obtain external funding to conduct their research. Indeed, their professional livelihoods depend on such funding.

lecture, I propose a way to find a balance between the desire to answer "what works" questions and the importance of developing deep understanding of the phenomena we study.

The Roles of Theory and Philosophy

Although MER was aptly characterized 15 years ago by Kilpatrick (1992) as largely *atheoretical*, a perusal of recent articles in major MER journals reveals that references to theory are commonplace. In fact, Silver and Herbst (2004) have noted that expressions such as "theory-based," "theoretical framework," and "theorizing" are commonly used by reviewers of manuscripts submitted for publication in the *Journal for Research in Mathematics Education* during the past four or five years. Silver and Herbst insist that manuscripts are often rejected for being atheoretical. I suspect the same is true of proposals submitted to other MER journals. In addition, although the direct role that philosophy should play in MER is less clear, it certainly is the case that researchers in most other behavioral and social sciences – e.g., anthropology, psychology, sociology – typically can articulate what their core beliefs and perspectives are about the phenomena they study. Moreover, these core beliefs and perspectives influence to methods they use to study these phenomena. Is the same true of researchers in mathematics education?

But, why should our research be based in theory and why should one care about our philosophical stance? In what follows, I argue that the role of theory should be determined in light of the research framework one has adopted and that this framework directly influences that research methods we use. Before proceeding further, however, let me first discuss the broader notion of *research framework* and then situate the role of theory within this notion. Then, I will discuss how the theoretical and philosophical perspectives we adopt influence our work.

The Role of Theory within a Research Framework

The notion of a research framework is central to every field of inquiry, but at the same time the development and use of frameworks may be the least understood aspect of the research process. The online *Encarta World English Dictionary* defines a framework as "a set of ideas, principles, agreements, or rules that provides the basis or the outline for something that is more fully developed at a later stage." I also like to think of a framework as being like a scaffold erected to make it possible for repairs to be made on a building. A scaffold encloses the building and enables workers to reach otherwise inaccessible portions of it. Thus, a research framework is a basic structure of the ideas (i.e., abstractions and relationships) that serve as the basis for a phenomenon that is to be investigated. These abstractions and the (assumed) interrelationships among them represent the relevant features of the phenomenon as determined by the research perspective that has been adopted.[4] The abstractions and interrelationships are then used as the basis and justification for all aspects of the research. A *theoretical* research framework – that is, a framework based on theory – does more than provide a structure for ideas. In particular, it serves at least four important purposes.

First, *a theoretical framework provides a structure for conceptualizing and designing research studies.* In particular, a theoretical framework helps determine:

- the nature of the questions asked;
- the manner in which questions are formulated;
- the way the concepts, constructs, and processes of the research are defined; and
- the principles of discovery and justification allowed for creating new "knowledge" about the topic under study (this refers to acceptable research methods).

Second, *there is no data without a theory to guide the researcher in making sense of those data.* We have all heard the claim, "The data speak for themselves!" Dylan Wiliam and I have argued that data actually have nothing to say. Whether or not a set of data can count as evidence of something is determined by the researcher's assumptions and beliefs as well as the context in which it was gathered (Lester & Wiliam, 2002; Wiliam & Lester, in press). One important aspect of a researcher's beliefs is the theory he or she is using; this theory makes it possible to make sense of a set of data.

Third, a good theoretical framework allows us to transcend common sense. Andy diSessa (1991) has argued that theory building is the linchpin in spurring practical progress. He notes that you don't need theory for many everyday problems—purely empirical approaches often are enough. But often things aren't so easy. Deep understanding that comes from concern for theory building is often essential to deal with truly important problems. Researchers should always endeavor to deeply understand the phenomena they are studying—the important, big questions (e.g., What does it mean to understand a concept? What is the teacher's role in instruction?)—not simply find solutions to immediate problems and dilemmas (i.e., determine "what works."). A theoretical framework helps us develop deep understanding by providing a structure for designing research studies, interpreting data resulting from those studies, and drawing conclusions.

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[4] By "perspective" I mean the viewpoint the researcher chooses to use to conceptualize and conduct the research. There are various kinds of perspectives: discipline-based (e.g., anthropology, psychology), practice-oriented (e.g., formative vs. summative evaluation), philosophical (e.g., positivist, interpretivist, critical theorist), etc.

Fourth, a theoretical framework serves as a structure of justification, *in addition to being a structure of explanation*. In my view, although explanation is an essential part of the research process, too often educational researchers are concerned with coming up with good "explanations" but not concerned enough with justifying why they are doing what they are doing and why their explanations and interpretations are reasonable. In my experience reviewing manuscripts for publication and advising doctoral students about their dissertations, I have found a lack of attention to clarifying and justifying why a particular question is proposed to be studied in a particular way and why certain factors (e.g., concepts, behaviors, attitudes, societal forces) are more important than others. A theoretical framework provides, then, an argument that the concepts chosen for investigation, and any anticipated relationships among them, will be appropriate and useful given the research problem under investigation. It also guides us in our choice of research methods.

The development of theory is absolutely essential in order for significant advances to be made in the collective and individual thinking of the MER community. But, not everything we know can be collapsed into a single theory. For example, models of realistic, complex situations typically draw on a variety of theories. Furthermore, solutions to realistic, complex problems usually need to draw on ideas from more than a single mathematics topic or even a single discipline. So, a grand “theory of everything” cannot ever be developed and efforts to develop one are very likely to keep us from making progress toward the goals of our work. Instead, we should focus our efforts on using smaller, more focused theories and models of teaching, learning and development. I propose that we view the frameworks we adopt for our research as sources of ideas that we can appropriate and modify for our purposes as mathematics educators. This process is quite similar to the thinking process characterized by the French word *bricolage,* a notion borrowed by both Cobb (2007) and Gravemeijer (1994) from Claude Levi–Strauss to describe the process of instructional design. A *bricoleur* is a handyman who uses whatever tools are available to come up with solutions to everyday problems. In like manner, we should appropriate whatever theories and perspectives are available in our pursuit of answers to our research questions. (In a later section I say more about the process of *bricolage*.)

The Place of Philosophy in MER

Discussions and debates over philosophical issues associated with MER are common (e.g., Cobb, 1995; Davis, Maher, & Noddings, 1990; Lesh & Doerr, 2003; Orton, 1995; Simon, 1995; Steffe & Thompson, 2000). Also, in a paper written for the *Second Handbook of Research on Mathematics Teaching and Learning*, Cobb (2007) puts "philosophy to work by drawing on the analyses of a number of thinkers who have grappled with the thorny problem of making reasoned decisions about competing theoretical perspectives" (p. 3). He uses the work of noted philosophers such as (alphabetically) John Dewey, Paul Feyerabend, Thomas Kuhn, Imre Lakatos, Stephen Pepper, Michael Polanyi, Karl Popper, Hilary Putnam, W. V. Quine, Richard Rorty, Ernst von Glasersfeld, and others to build a convincing case for considering the various theoretical perspectives being used today as sources of ideas to be used or adapted to the purposes of mathematics education researchers. Cobb observes that –

> The theoretical perspectives ... include radical constructivism, sociocultural theory, symbolic interactionism, distributed cognition, information-processing psychology, situated cognition, critical theory, critical race theory, and discourse theory. ... Proponents of various perspectives frequently advocate their viewpoints with what can only be described as ideological fervor .... In the face of this sometimes bewildering array of theoretical alternatives ... I seek to address ... how we might make and justify our decisions to adopt one theoretical perspective rather than another (p. 3).

Cobb (2007) goes on to contrast four theoretical perspectives by considering the manner in which these perspectives orient and constrain the types of questions asked about the learning and teaching of mathematics, the nature of the phenomena investigated, and the forms of knowledge produced. (Table 1 displays a summary of this analysis).

In a similar way, Dylan Wiliam and I (Lester & Wiliam, 2002; Wiliam & Lester, in press) have demonstrated how philosophy can help us determine what counts as evidence in MER. Specifically, we borrowed a classification of systems of inquiry developed by Churchman (1971). This scheme classifies all systems of inquiry into five broad categories, each of which he labeled with the name of a philosopher (viz., Leibniz, Locke, Kant, Hegel, and Singer) that best exemplifies the stance involved in adopting the system. Churchman gave particular attention in his classification to what is to be regarded as the primary or most salient form of evidence. We found Churchman's framework to be particularly useful in thinking about how to conduct research that makes a difference, and specifically, whether the research moves people to appropriate action.

The foregoing analyses illustrate how orienting our research activities in philosophy can enhance the quality of our work and, ultimately, how it can serve both theoretical and practical ends.

Table 1

Contrasts among 4 Theoretical Perspectives (Cobb, 2007, p.28)

Theoretical perspective	**Characteristics of the individual**	**Usefulness**	**Limitations**
Experimental psychology	Statistically constructed collective individual	Administration of educational systems	• Limited relevance to design at classroom level
Cognitive psychology	Epistemic individual as reorganizer of activity	• Specification of "big ideas" • Design of instructional activities	• Means of supporting learning limited to instructional tasks
Sociocultural theory	Individual as participant in cultural practices	• Designs that take account of students' out-of-school practices • Designs that take account of the institutional setting of teaching and learning	• Limited relevance to design at classroom level
Distributed cognition	Individual element of a reasoning system	• Design of classroom learning environments including norms, discourse, and tools	• Delegitimizes cognitive analyses of specific students' reasoning

The Purposes of MER

In his book, *Pasteur's Quadrant: Basic Science and Technological Innovation,* Donald Stokes (1997) presented a new way to think about scientific and technological research and their purposes. Because certain of his ideas have direct relevance for MER and the roles of theory and philosophy, let me give a very brief overview of what he proposed.

Stokes began with a detailed discussion of the history of the development of the current U.S. policy for supporting advanced scientific study (I suspect similar policies exist in other industrialized countries). He noted that from the beginning of the development of this policy shortly after World War II there has been an inherent tension between the pursuit of *fundamental understanding* and *considerations of use* (these are his terms). This tension is manifest in the often-radical separation between basic and applied science. He argued that prior to the latter part of the 19th Century, scientific research was conducted largely in pursuit of deep understanding of the world. But, the rise of microbiology in the late 19th Century brought with it a concern for putting scientific understanding to practical use. Stokes illustrated this concern with the work of Louis Pasteur. Of course, Pasteur working in his laboratory wanted to understand the process of disease at the most basic level, but he also wanted that understanding to be applicable to dealing with, for example, anthrax in sheep and cattle, cholera in chickens, spoilage of milk, and rabies in people. It is clear that Pasteur was concerned with both fundamental understanding and considerations of use.

Stokes proposed a way to think about scientific research that blends the two motives: the *quest for fundamental understanding* and *considerations of use*. He depicted this blending as shown in Figure 1, where the vertical axis represents the quest for fundamental understanding and the horizontal axis considerations of use. So, Pasteur's research belongs in the upper right quadrant, but what of the other three quadrants of the figure? Consider first the upper left quadrant. Neils Bohr came up with a radical model of the atom, which had electrons orbiting around a nucleus. Bohr was interested solely in understanding the structure of the atom; he was not concerned about the usefulness of his work. Research in the lower right quadrant is represented by the work of Thomas Edison on electric lighting. Edison was concerned primarily with immediate applicability; his research was narrowly targeted, with little concern about deeper implications or understanding. (It may be that Edison's lack of interest in seeking fundamental understanding explains why he did not receive a Nobel Prize.[5]) Finally, in the lower left quadrant we have research that involves explorations of phenomena without having in view either explanatory goals or uses to which the results can be put. One would hope that little, if any, research has taken place in science, education, or any other field in this

[5] The director of the Institute for Education Sciences of the U.S. Department of Education, R. J. Whitehurst (2003) has insisted that "the model that Edison provides of an invention factory that moves from inspiration through lab research to trials of effectiveness to promotion and finally to distribution and product support is particularly applicable to education . . . . randomized trials are the only sure method for determining the effectiveness of education programs and practices" (p. 4). So, it seems that not every considers Edison's lack of concern for understanding problematic.

quadrant—no interest in fundamental understand or consideration of usefulness)—but I suspect that such research has been conducted.

Stokes then presented a somewhat different model (he referred to it as a “revised, dynamic model,” p. 88) for thinking about scientific and technological research. In this model, the outcome of pure, basic research is still an increase in understanding and the outcome of pure, applied research is an improvement over existing technology. By melding the two types of research, we get *use-inspired, basic research* that has as its goals increased understanding *and* technological advancement. Adapting Stokes’s dynamic model to educational research in general, and MER in particular, I have come up with a slightly different model (see Figure 2). There are two minor, but important differences between my model and Stokes’s. First, I have broadened pure, applied research to include “development” activities. Second, I have substituted “technology” with “products” (e.g., instructional materials, including curricula, professional development programs, and district educational policies).

Figure 1. Stokes’s (1997) model of scientific research

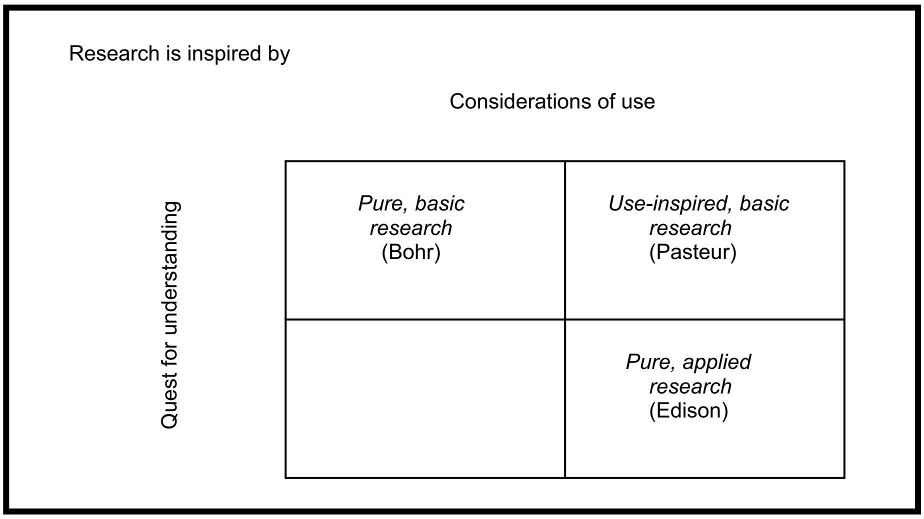
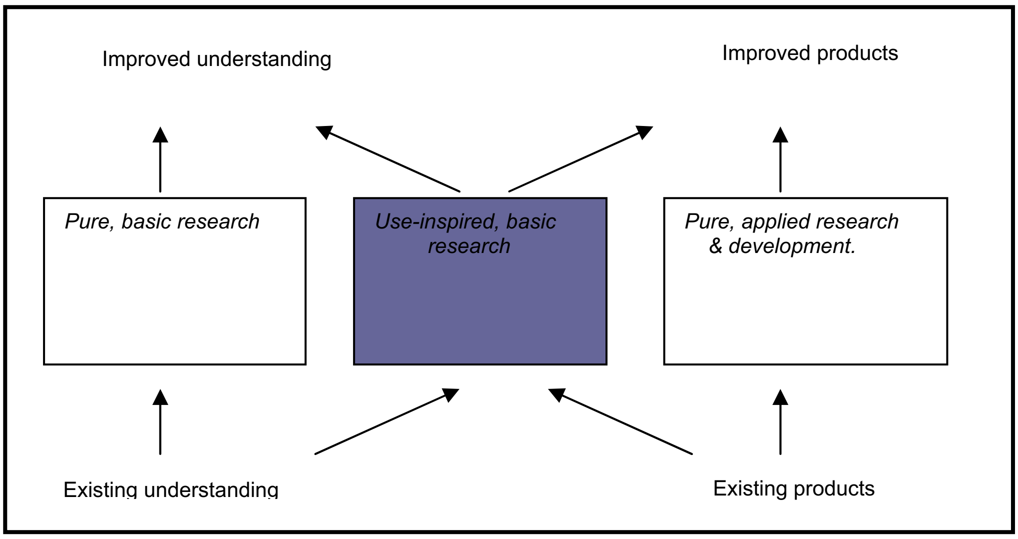


Figure 2. Adaptation of Stokes’s “dynamic” model to educational research.



I suggest, as do Cobb (2007) and Gravemeijer (1994), that rather than adhering to one particular theoretical perspective, we act as *bricoleurs* by adapting ideas from a range of theoretical sources to suit our goals—goals that should aim not only to deepen our fundamental understanding of mathematics learning and teaching, but also to aid us in providing practical wisdom about problems practitioners care about.

Types of Legitimate Research Activities

But, how should we pursue both our desire for fundamental understanding and the need to put our results into practice? This question suggests taking a look at the types of research activities we engage in. Just such a look was undertaken by the American Statistical Association (ASA).

Recently, I was a member of a working group of the ASA charged with the task of preparing a document that would contribute "to improving the quality of mathematics education research" (Working Group on Statistics in Mathematics Education [WGSME], 2006, p. 1). The working group included both statisticians and mathematics educators and had as its primary goal to show how and why mathematics education researchers might use statistical methods more effectively in their research. The working group noted that, although quantitative methods had been all but abandoned in recent years for qualitative methods, both quantitative and qualitative methods are necessary if research is to have any lasting impact on what we know about mathematics teaching and learning. The discussion that follows is based largely on the report of the WGSME and serves as a final lead-in to a discussion of research methods.

Key Features of the WGSME Report

Need for MER to be more cumulative. Two features of the WGSME report are particularly relevant to a discussion of research methods. First, the WGSME insisted that if research in mathematics education is to influence practice, it must become more cumulative in nature; new research needs to build on existing research to produce a more coherent body of work. In addition, although researchers in mathematics education should be free to pursue the problems and questions that interest them, in order for this work to influence practice it must be situated within a larger corpus. Now here is the interesting, relevant part:

> School mathematics is an excellent venue for *small-scale studies* because mathematics learning has many facets, and the classroom is a manageable unit that can be studied in depth and detail. Such studies can cumulate, however, only if they are connected. Studies cannot be linked together well unless researchers are consistent in their use of interventions; observation and measurement tools; and techniques of data collection,

> data analysis, and reporting. . . . And as Raudenbush (2005) points out, a well-integrated research effort demands *methodological diversity*. These guidelines are offered with a goal of promoting opportunities for mathematics education research to have a collective impact. (WGSME, 2006, pp. 4-5, italics added for emphasis)

It is notable that the ASA, a group devoted to the use of statistical methodologies in all types of research, recognized the value of *small-scale studies*, which typically do not depend upon quantitative methods, and the importance of *methodological diversity* in our work.

MER should be situated within a larger research program. The second key feature of the WGSME report has to do with the nature of research programs in mathematics education. The authors of the report insist that:

> Because there are so many possible classifications, the guidelines of this report are organized within the framework of a *research program* rather than an individual study. An individual study may not possess all the components of a research program, but it can certainly be situated somewhere within the framework. (WGSME, 2006, p. 5).

Figure 3, taken from the report, illustrates how a comprehensive research program might be structured. (For simplicity, the figure displays the main components in a linear fashion, with the understanding that in an actual research program those components would be mutually interactive and cyclic. As one of the authors of the report, I have taken the liberty of including a verbatim excerpt from it that discusses the structure and components of a research program.

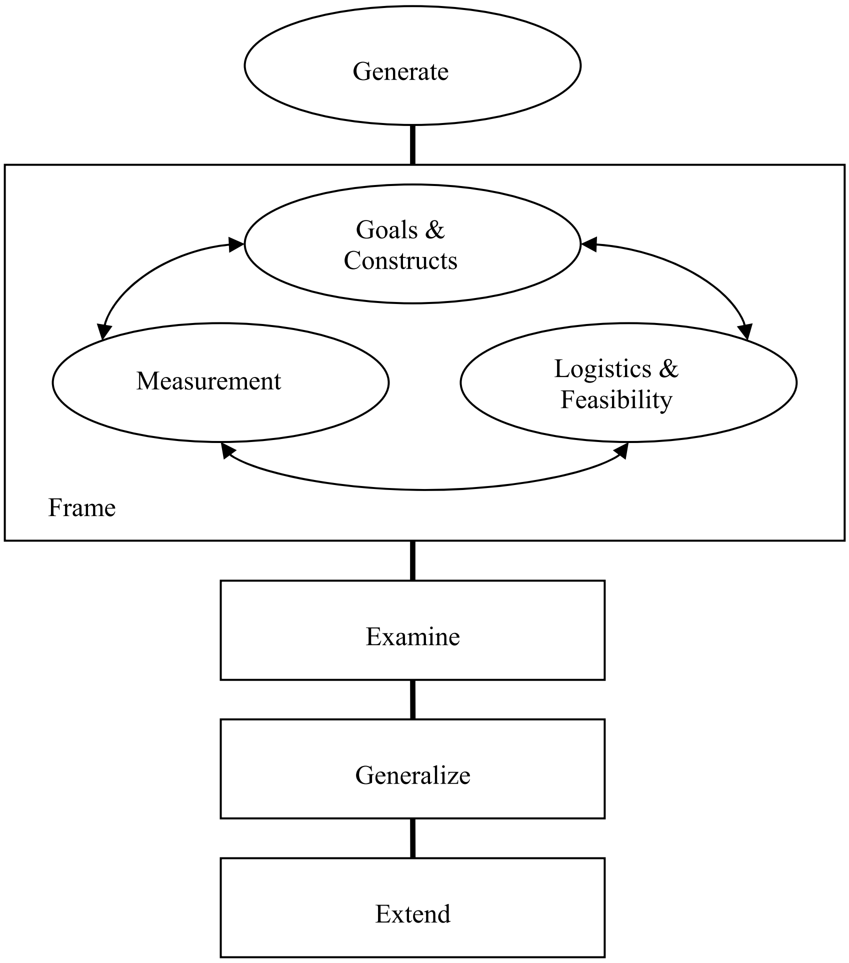
> The first component—although not necessarily the point at which a given single research study would begin—is to *generate*. To launch a research program, mathematics educators need to generate some ideas about the phenomena of interest so that they can begin to explore them. Those ideas might emerge from theoretical considerations, previous research, or observations of practice. Research performed within this component might be analytic and not empirical, but it might also involve exploring an existing data set or analyzing a set of research studies in order to yield new insights.
>
> Once ideas have been generated, they need to be *framed* in some fashion. A frame is seen as involving clarification of the *goals* of the research program and definition of the *constructs* it entails, formulation of tools and procedures for the *measurement* of those constructs, and consideration of the *logistics* needed to put the ideas into practice and study their *feasibility*. The components within a frame are developed interactively as researchers decide what the program's initial research questions or hypotheses should be and how studies might best be shaped and managed. Researchers begin exploratory empirical studies of the phenomena of interest. They might try out a proposed intervention to see if and how it works, or they might develop and test an instrument to measure a construct of interest.

> Framing the domain of study leads researchers to *examine* the phenomena more systematically. Research studies within this component are necessarily restricted in scale, whether in effort, time, or human and material resources. Their purpose is to understand the phenomena better and to get indicators of what might work under which conditions. The results of these studies may cycle back to the frame component, yielding modifications in constructs, instruments, research plans, or research questions. Or, if sufficiently robust, the results may lead to the next component.
>
> Once small-scale research studies have examined phenomena through observation or intervention, more-comprehensive studies can be mounted that seek to *generalize* what has been found. That generalization can address questions of scale (studying different populations or sites, using more comprehensive measures, examining different implementation conditions), or it can be used to refine the theory or reframe the entire research program.
>
> A body of research that has yielded some generalizable outcomes can be *extended* in a variety of ways. Multiple studies can be synthesized; long-term effects can be examined; policies can be developed for effective implementation. Follow-up research studies can refine the instruments and procedures, include a greater variety of participants, and generalize the findings of earlier work still further.
>
> It is essential to understand that each component of the research model of Figure 1 [*Figure 3 in this paper*] has the possibility and potential to cycle back to any earlier component, and such cycling should be a conscious effort of the researchers. Consequently, progress in research is generally more of a circular process than a linear one. (WGSME, 2006, pp. 5-7).

The report goes on to provide guidelines for conducting research within each component of the research program as well as characteristic activities and illustrative research scenarios for the components. What is important for purposes of this paper is that experimental, strictly quantitative methods are called for only for the "generalize" and "extend" components. For the other components, other methods (e.g., case studies, ethnographies, in-depth interviews, design experiments) are recommended.

In the next section, I describe one recommended type of research method: design experiments.

Figure 3. Structure and components of a research program (WGSME, 2006, p. 6).**Error!**



The Promise of the Design Experiment as a Research Method

The discussions in the preceding sections serve as a prelude to what follows to the extent to which they have established the need to consider the methods we use in MER with respect to the theory frameworks we employ, the philosophical perspectives we adopt, and the purposes of our work. In this section I turn my attention to establishing the importance of a particular set of research methods – namely those situated within a design research perspective. But, before beginning the discussion of methods, I should point out that there is at least one other ingredient in deciding upon the methods to employ: context.

Consider the challenge of conducting meaningful research on the implementation of a new (reform) mathematics curriculum.[6] Before deciding on the research methods that are likely to

[6] Since the early-1990s there have been no fewer than 15 large-scale, federally-funded mathematics curriculum development projects; the largest infusion of federal funds into school mathematics since the new math era of the 1950s and 60s. One result of this expenditure of money has been an ever-growing interest in determining the effectiveness of these new curricula.

be most useful in undertaking such research, the researcher must also take into account issues of purpose and context. Wiliam and Lester (in press) capture some of the complexity of the situation:

> The current controversy over reform versus traditional mathematics curricula has attracted a great deal of attention in the United States and elsewhere among educators, professional mathematicians, politicians, and parents. . . . For some, the issue of whether the traditional or reform curricula provide the most appropriate means of developing mathematical competence is an issue that can be settled on the basis of logical argument. On one side, the proponents of reform curricula might argue that a school mathematics curriculum should resemble the activities of mathematicians, with a focus on the *processes* of mathematics. On the other side, the anti-reform movement might argue that the best preparation in mathematics is one based on skills and procedures.

Wiliam and Lester go on to describe a scenario that further complicates the situation:

> A team of researchers, composed of the authors of a reform-minded mathematics curriculum and classroom teachers interested in using that curriculum, decide after considerable discussion and reflection to design a study in which grade 9 students are randomly assigned either to classrooms that will use the new curriculum or to those that will use the traditional curriculum. The research team's goal is to investigate the effectiveness (with respect to student learning) of the two curricula over the course of the entire school year. Suppose further that the research design they developed is appropriate for the sort of research they are intending to conduct. From the data the team will gather, they hope to be able to develop a reasonable account of the effectiveness of the two curricula, relative to whatever criteria are agreed upon, and this account could lead them to draw certain conclusions.

A major difficulty with the research design the researchers have decided to use is that, because observations are regarded as evidence, it is necessary for all observers to agree on what they have observed. Because what researchers observe is based on the theories and epistemological perspectives they have, different people will observe different things, even in the same classroom, and different conclusions will be drawn.

The situation presented in Wiliam and Lester's scenario becomes even more complicated once the evidence has been gathered:

> After studying the evidence obtained from the study, the research team has concluded that the reform curriculum is more effective for grade 9 students. . . . However, a sizable group of parents strongly opposes the new curriculum. Their concerns stem from beliefs that the new curriculum engenders low expectations among students, de-emphasizes basic skills, and places little attention on getting correct answers to problems. The views of this group of parents, who happen to be very active in school-related affairs, have been influenced by newspaper and news magazine reports raising questions about the new curricula, called "fuzzy math" by some pundits. To complicate matters further, although the teachers in the study were "true believers" in the new curriculum, many of the other mathematics teachers in the school district have little or no enthusiasm about changing their traditional instructional practices or using different materials, and only a few teachers have had any professional development training in the implementation of the new curriculum.

The situation as portrayed by Wiliam and Lester suggests that conducting research with real students, in real classrooms, in real schools is so complex as to almost defy analysis using methods found to be so successful in fields of study such as the natural sciences and agricultural. Indeed, it appears that research associated with the *generalize* and *extend* components of the WGSME (2006) research program that relies on traditional, quasi-experimental methods may be nearly impossible to conduct in a way that will actually inform practice in classrooms. So, what sorts of research methods are likely to afford meaningful results? The answer lies in the methodologies of design research. Another bit of history is in order before discussing design research.

In their history of the maturation of MER in the United States, Lester and Lambdin (2003) noted that by the early 1990s the growing interdisciplinarity of MER and the corresponding explosion of research methodologies were seen by some as indicators that researchers had begun to recognize that the research methods of the past were simply inadequate for dealing with the questions and issues that they now regarded as important. During the first half of the 1990s, concern about the very nature of MER activity was beginning to be discussed. For example, early in my term (1992-1996) as editor of the *Journal for Research in Mathematics Education* (JRME), my editorial board had extensive discussions about difficulties in evaluating research that employed methods borrowed from other disciplines. These difficulties often stemmed from lack of regard, on the part of researchers, for the standards established within those disciplines for use of those methods.

At about the same time, similar concerns were developing among the staff of the U.S. National Science Foundation (NSF) program Research in Teaching and Learning (RTL). This concern led the NSF to fund conferences in 1994 and 1995 whose goals were to identify the research methods most useful in answering important questions in mathematics and science education research and then to clearly delineate the appropriate uses of those methods. Barbara Lovitts and Richard Lesh, then RTL Program Directors at NSF, provided the following background prior to the first of the two conferences:

> Over the last few years [we] have noticed a persistent problem in preliminary and full proposals sent to the RTL program: the thoughtfulness about proposal methodology from top to bottom (i.e., questions, hypotheses, data collection, reduction, and analysis), in general, is often inadequate. . . . This inadequacy, we believe, stems in part from the newness of methodologies fruitfully being employed in science and mathematics education research and from lack of any deep and serious consideration of, or attempt at, standardization or codification of the methodologies. (Barbara Lovitts, e-mail correspondence to potential conference participants, 18 November 1994)

Lovitts and Lesh also provided some historical context and set the goals for the conferences:

> [We] hope that one thing that people who attend the [first] Santa Fe Conference [will do] is to think about their own research and to ask themselves: (1) What kind of innovative methodologies do I expect to need to use in the next few years? (2) What criticisms can I expect concerning the appropriateness of this methodology? (3) How will colleagues be able to distinguish the high quality of my stuff from the low quality stuff produced [by someone else]? (4) What would I need to tell a capable graduate student so that he

> or she could replicate my results, or assist me at a remote site? (Richard Lesh, e-mail correspondence to potential participants, 2 November 1994)

The Santa Fe conferences, attended by some of the most prominent American mathematics and science education researchers, and concomitant e-mail correspondence involved very lively discussion and debate about the philosophy, epistemology, and methodology of research in mathematics and science education, and raised questions about the very purpose of research in these two evolving fields. In fact, in a way, and to an extent that had rarely occurred before, participants came away from these conferences in general agreement on several key issues about mathematics and science education research, among them the following:

1. There is a need to develop new methods for certain research purposes.
2. The methodology a researcher uses is an expression of the researcher's "interpretive stance"[7] with respect to mathematics/science content, research purposes, and the nature of knowing and learning.
3. Methodological considerations should not be separated from considerations of how we come to know, which in turn are based on our most fundamental beliefs and principles. (From my notes taken at the conferences and from e-mail correspondence with various conference participants, 15 November 1994 – 3 December 1995.)

An important outcome of these conferences was the publication of an edited volume dealing with research issues and focusing on innovative research designs and methods (Kelly & Lesh, 2000). But, in my view, the most notable aspect of the conferences and the resulting volume was the almost universal agreement among those participating in this project on the need for new research methods to address questions mathematics and science educators care about. A quick perusal of the chapters in this volume might suggest that all manner of research methods were being promoted: teaching experiments, classroom-based research, clinical methods (e.g., task-based interviews, videotape analysis), curriculum design, and assessment design. A closer look, however, reveals some important commonalities: all can be classified as exemplifying principles of “design experiments” or “design research.”

The term “design experiment” was introduced to education researchers by Ann Brown (1992) and Alan Collins (1992). Although design experiment methods are relatively new in

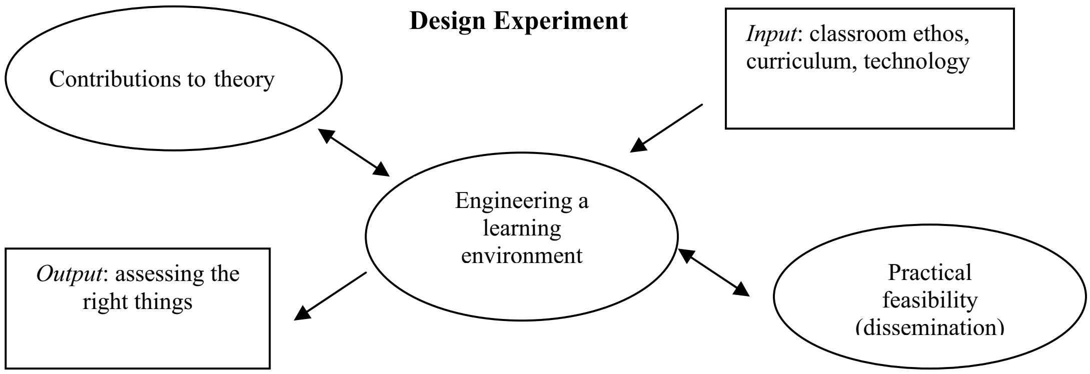
[7] Conference participants agreed that a researcher’s “interpretive stance” was his or her set of beliefs and assumptions about the object of the research, the nature of the generalizations the researcher wants to make, and the nature of mathematics/science (From my conference notes, 1994-95).

educational research, they have been used productively for several years in such fields as architecture and engineering. In these fields, design experiments have four primary distinguishing characteristics:

1. *They should involve designing some complex artifact or conceptual tool* – in education this might include software or some other kind of instructional materials for learning, assessment, or problem solving.
2. *The design process involves the development of a construct, or model and this construct or model often is the primary result that the research is intended to produce.* For example, when a person designs something, one of the most significant parts of the product may be the underlying design itself. Or, when a person constructs something, one of the most significant parts of the product may be the underlying construct.
3. *The product that is produced is needed for some specific purpose(s), situation(s), and client(s); and, these purposes provide objective criteria for assessing the quality of the underlying construct or model.*
4. *The design process involves experiments that usually occur during a series of iterative cycles (development → testing → revision) that often use both qualitative and/or quantitative forms of feedback.* These design cycles automatically generate auditable trails of documentation whose trajectories reveal important information about the nature of developments that occur concerning artifacts and underlying conceptual systems. (Lesh & Kelly, 2002)

Schoenfeld (2007) explains that for researchers such as Brown and Collins, design experiments were exercises in instructional *design*, involving the creation of instructional environments (contexts and materials) that were realized as real instruction. They also were true *experiments* – not necessarily in the statistical sense, but in the scientific sense, in that theory-based hypotheses were made, measurement tools were established, a deliberate procedure was rigorously employed, outcomes were measured and analyzed, and theory was revisited in view of the data. Figure 4 is Brown's (1992) attempt to capture this complexity. Note that the double arrow between "contributions to learning theory" and "engineering a learning environment" represents the dualism inherent in working in Pasteur's quadrant (refer to my earlier discussion in the section "Purposes of MER").

Figure 4. Features of design experiments (Schoenfeld, 2007, p. 98, who reprinted it from Brown, 1992, p. 142)



Schoenfeld (2007) points out that the meaning of design experiments has not been "settled" in the literature. Related but clearly distinct formulations may be found in the work of various researchers, among them: Ball (2000), Battista and Clements (2000), Cobb, Confrey, diSessa, Lehrer, and Schauble (2003), Collins (1999), the Design-Based Research Collaborative (2003), Gravemeijer (1994), Lesh and Doerr (2003), and Steffe and Thompson (2000). Despite the differences in emphasis among those who theorize about and use design experiments, there is certainly an agreement that an important outcome of design experiments is a tested design for a curriculum or instructional intervention that has promise in terms of student learning (Schoenfeld, 2007). This should be, after all, a primary goal of all MER: the development of improved products, as well as an increase in understanding (see Figure 2).

A Personal Reflection

I wish to bring this lecture to a close by adding a personal reflection as a mathematics educator who has been in the business for more than 35 year about the state of our research. Although the tone of some of my remarks have been somewhat negative, I do not want to leave you with the impression that MER is in a sorry state because I simply do not believe this to be the case. When I began doing research in mathematics education in the early1970s statistical methods dominated the field – at least in the United States – but, as the quote from Scandura (1967) cited at the beginning of this lecture suggests, the use of these methods was too often unsophisticated and wrongheaded. Moreover, we typically looked to psychologists, many of who knew very little mathematics and thought little, if at all, about the sorts of questions mathematics educators care about for direction in determining their research programs. This has changed! Indeed, the progress that has been made during the course of my career has been remarkable. There have been major changes in the way we think about and conduct our practice as researchers. Put simply, we have developed our own identity as a research community – the very existence of PME attests to this, as does the healthy state of the several research journals that have come into existence during the past 40 years (see Lester & Lambdin [2003] for a history of the maturation of the MER community).

The changes in the types of research methods we use serve to illustrate just how dramatic the changes have been. In 1973, when the *Journal for Research in Mathematics Education* (JRME) had been in existence for three years, articles published in JRME were overwhelmingly statistical in nature. Ten years later, in 1983, about one-third of the articles in JRME were non-statistical in nature, and by 1993, well over one-half (62%) used methods other than statistics (Lambdin & Kloosterman, 1995). Data for 1994-95 indicated that only about 14 percent of the research studies published in *JRME* made primary use of inferential statistical analysis. Today, we find that the methods we use in our research cover a very wide range of types and represent several quite different theoretical and philosophical perspectives. I see these changes as indicators of a healthy community; a community that embraces diversity of ideas and approaches

The shift in research methods represents a much more profound change than a simple abandoning of traditional approaches in favor of contemporary methods. Instead, it indicates both the adoption of a broader conception of research in the field and a significant departure from an empiricist, positivistic paradigm toward more narrative, interpretive perspectives. I expect the MER community to turn now to paying more attention to how our research can make lasting changes in both policy and practice, while at the same time continuing to develop rich, robust theories of mathematics learning, teaching, and development.

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WHAT CONSTITUTES GOOD PRACTICE IN TEACHING MATHEMATICS, A PERSONAL PERSPECTIVE

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Abstract: The aim of this article is to look into the concept of good practice in the teaching of mathematics. I will present my idea of good practice through the formulation of seven principles which will guide the teacher's activity in the classroom. Two extensive illustrations of Grade 8 teaching will be presented to illuminate the seven principles. A way to put together video-recording and the teacher's written reflection will be suggested as a way of inferring the teacher's possible values. Advantages and deficiencies of the analysis of video recordings will be discussed.

Mottos

What we have to learn to do, we learn by doing. (Aristotle)

Spoon feeding in the long run teaches us nothing but the shape of the spoon. (E.M. Foster)

The art of teaching has little to do with the traffic of knowledge, its fundamental purpose must be to foster the art of learning. (Glasersfeld, 1997, p. 192)

INTRODUCTION

Two years ago I started a seminar called 'Chapters in Mathematics Education', aimed at future mathematics teachers. Similar to many other courses abroad, I decided to use video analysis as the main tool of the seminar. For this purpose, I acquired videos from the TIMSS 1999 Video Study. When I was thinking about the ways students could analyse the videos and the strategies I could use to focus their attention on some aspects of teaching, I realised that I myself should first formulate what I considered to be good practice in teaching, that is I needed to uncover my underlying notions about what constitutes good mathematics teaching. I then strove to put into words something which is actually very elusive. In this article, I want to share with you the results of my efforts.

My views of what constitutes good practice in the teaching of mathematics is culturally affected by my experience from primary and secondary schools and my university studies and above all, I believe, by my interaction with colleagues at the faculty where I teach. Moreover, I cannot omit constructivism and its influence.

Before I started to teach mathematics, I had never been exposed to any teaching which might be termed 'discovery' or 'constructivist'. The lessons I experienced were mostly well structured but the content was delivered by transmission methods. I was a good pupil, I always learned what I was told without much speculating about it.

I enjoyed mathematics for its own sake and somehow learnt to grasp most of what we did without actually being given time and space to 'discover' it.

When I started teaching mathematics at the university, I was lucky to be appointed to work with Milan Hejny as a teacher leading the seminars in his courses. It was a different world for me! It was as if I had to re-learn mathematics all over again and I suddenly could see that mathematics could be much more interesting than I had previously thought. Except for the teaching itself, I was also given some long term mathematical work in which I had to solve a series of problems. Some of them were left unsolved for a long time. During this time, Milan was giving me a lot of support but not many hints! Thus, I learned about principles of constructivism applied in practice long before I actually read about constructivism itself.

There are many different understandings of constructivism. The word is often preceded by attributes such as social, radical, pedagogical, didactic. [1] What all the different approaches to constructivism have in common is an emphasis on the mathematical activity of pupils:

> Learning mathematics requires construction, not passive reception, and to know mathematics requires constructive work with mathematical objects in a mathematical community. (Davis, Maher & Noddings, Eds., 1990, p. 2)

As constructivism is a theory of learning, not teaching, there have been many attempts to create "bridging theories" which link constructivism to the practice of schools [2].

> As a theory, constructivism is intended to offer insight into human learning, and it presents learning as a truly complex phenomenon that is subject to an array of subtle and imposing, explicit and tacit, deliberate and accidental, social and biological influences. Such an account says little about what a teacher must do, although it does have something to say about what a teacher cannot do. In particular, a teacher cannot control learning. (Towers & Davis, 2002)

I believe that in a way, even our understanding of the potential implications of constructivism for the classroom, is our *personal construct*. My position will be clearer from the rest of the article.

I am not radical in the sense that everything should be discovered. In accord with Kurina, I also want to give space to the teacher to equip pupils with some knowledge as ready-made provided this is done "in the service of the emerging world of mathematics in the pupil's mind" (Kurina, 2002a). The teacher can provide pupils with necessary knowledge as the building block of the new construction and/or with some instructions of how to solve partial problems in order to enable them to get to a 'bigger' discovery. For example, if a pupil wants to explore a non-convex quadrilateral and cannot remember how to calculate the area of a triangle, then the teacher can remind him/her of the formula (or better, he/she can ask other children whether they know it) in order he/she would not lose track of the problem being solved.

Thus transmission and construction are not necessarily in opposition but rather complement each other. Moreover, some knowledge cannot be discovered, we also need information to be able to learn. For example, percent is marked by the sign %. But what it means or where and how it can be used can originate in the pupil's mind by his/her activity.

I often have discussions with my students, future mathematics teachers, about this apparent dichotomy of transmission versus construction. [3] Understandably, they tend to be far more critical than in-service teachers when discussing a lesson on the video. They notice instances in which the teacher gives the pupil a solution without letting him/her discover it for themselves. The interpretation of such an event depends on the goal which the teacher has for the lesson. Pupils may have previously discovered some knowledge and the teacher only reminds them of it.

COMPLEX NATURE OF TEACHING

Teaching is a very complex notion and can be studied from different points of view: pupils and their learning, material used, problems used, social climate, communication, values and beliefs of pupils, of teachers, images of mathematics they hold, etc. Here, I will concentrate on *the teacher* and his/her actions in the classroom. His/her role is not an easy one. For example, Elbers (2003) emphasises teachers' double role. On one hand, they are "in charge and responsible for the students' activities. They decided what topics would be worked on and they have their own ideas of what knowledge students should acquire during the lessons." On the other hand, they want "the students to find out for themselves: to invent solutions to problems and to prove their validity" and do not want to "frustrate children's creativity by using their authority for supporting certain answers instead of others". (ibid) Similarly, Skott characterises the role of a (reform) teacher like this:

> In short, the teacher is required to manoeuvre independently and autonomously in order to sustain individual and collective learning opportunities through on-the-spot decision-making. ... The reform, then, primarily calls for a reflective approach on the part of the teacher. (Skott, 2004)

Because teaching is a complex process, it "is not possible to state clear criteria and indicators for good teaching which can be applied in a general way. Each school, each class, each teaching situation is unique, has its genuine context and thus needs specific norms." (Krainer, 2005) Good teaching is thus context dependent. For example, Cooney (2001) emphasises that "...activities or teaching methods in and of themselves are neither good nor bad. Rather, it is the context which makes them effective or not."

Even though no criteria of good teaching which would be generally accepted can be given, researchers, teachers and student teachers have their own ideas and assumptions what good teaching means. Studies have shown that for many in-service and pre-service mathematics teachers, good teaching mostly means to be able to explain things clearly and provide an environment free of anxiety.

> The notion that good teaching consists of a telling process embedded in a humanistic, caring environment, assumes that the learning process should avoid frustration and not place students in a cognitively challenging environment. (Cooney, 2001)

In the questionnaire which we give regularly to in-service teachers taking part in our courses, pupils' intellectual development is in most cases neglected. When asked what is the main role of teaching mathematics in the primary school, they mostly stress basic formulas, proficiency in calculations, etc., but not the development in thinking or the introduction to intellectual work. The content is stressed. Similar results can be found in Cachová's research (Cachová, 2003). Consider, for example, the following quote of a practising teacher:

> The main goal of teaching mathematics at the primary school is to provide basic information – addition, subtraction, multiplication, division. To teach basic formulas (perimeter, area, surface area, volume), teach pupils to be able to use their theoretical knowledge in practice.

PRINCIPLES FOR GOOD PRACTICE IN THE TEACHING OF MATHEMATICS

In this section, I will present my idea of good practice in the teaching of mathematics. This will be done through seven principles formulated from the point of the teacher and his/her activity in mathematics lessons. [4] Considering that I am speaking about such complex issues as teaching and mathematics classroom, it is impossible to avoid certain repetitions and find something really original.

The principles are not discrete, they are mutually interwoven and one cannot be observed or investigated without reference to the others (see Fig. 1). Naturally, the list cannot be complete, it could be widened.

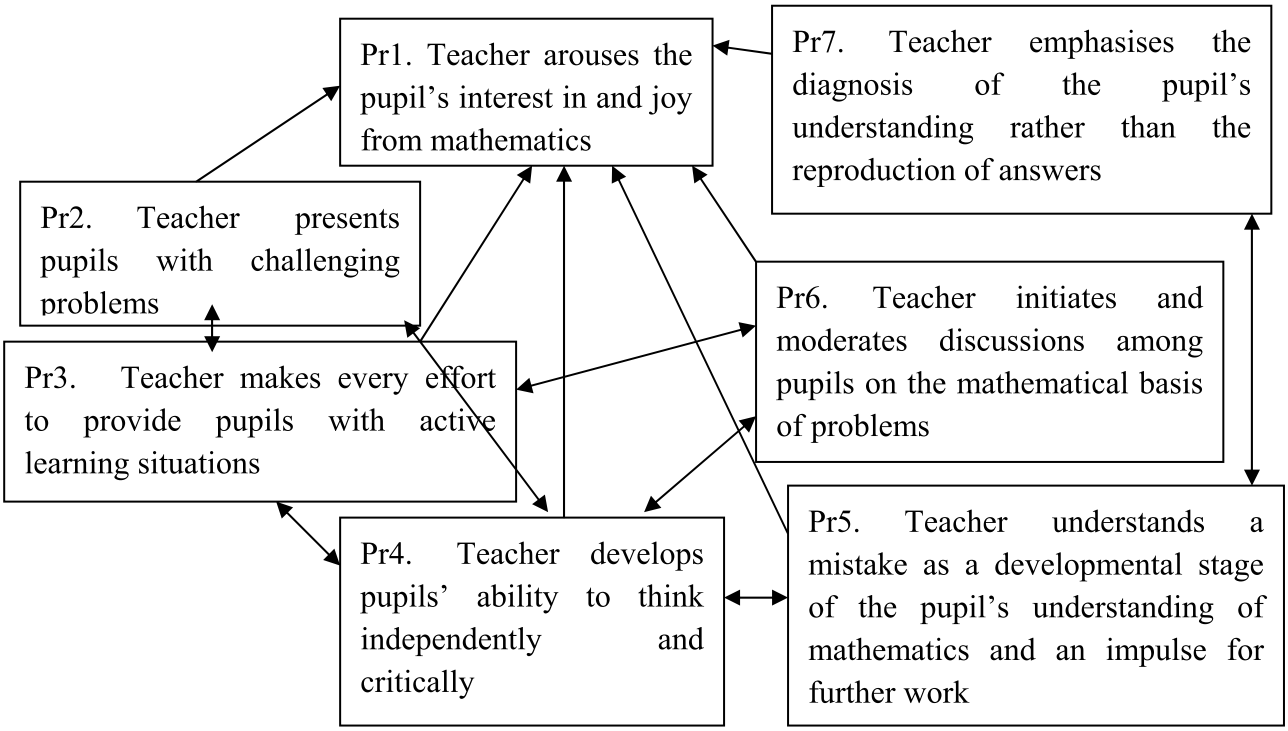

Fig. 1: A diagram of principles and their possible relationships

To each of the principles you may pose the questions *How?* or *What*? How does the teacher arouse the pupil's interest? How does he/she realise the third principle? What do active learning situations mean? etc. There is no single correct answer to any of these questions and moreover, the answers are context dependent. What will work with one teacher and one group of pupils may not work with another group. Fulfilling the above principles in one's teaching (whatever it might mean) in itself does not guarantee perfect results. I have on purpose neglected other important conditions, such as types of pupils we teach, their momentary disposition, classroom climate, school climate, influence of the family, of the society, teacher's teaching style, etc.

It goes without saying that in order for the teacher to be able to implement these principles in his/her teaching, he/she has to have sufficient knowledge of mathematics.

ILLUSTRATIONS AND DEFICIENCIES OF OUR COMMENTS ON THEM

In order to communicate what lies (for me) behind the principles, I will illustrate them by extracts of mathematics lessons and describe them in terms of some of the principles. The extracts come from TIMSS 1999 Video Study, from the videos sold by Lesson Lab, but I will not deliberately reveal which countries they come from. This is motivated by the lecture at CERME during which much of the discussion focused on the description of teaching in the two countries and their comparison which was not my aim.

Of course, it would be naive to suppose that what will be written (or what was seen on the video) is objective reality. There is no description without interpretation even when we have a video case. Not only was the lesson filmed by only one camera but my very choice of some extracts rather than others manifests a view of what I want the reader to notice about the illustration.

> What we notice is completely framed by what we know. ... An event of noticing is always and already an event of interpretation. (Towers & Davis, 2002)

Last but not least, I have to stress that I have *a great respect* for teachers who were willing to be video-recorded and gave their consent that their lesson could be used for educational purposes. Nobody's teaching would survive a detailed scrutiny without criticism (Clarke, 2001). Thus, even though it might be difficult, the reader should interpret my comments as such as I try to focus not on the teacher's teaching competence, but on the viability and appropriateness of his particular act (both teachers being male) which takes place in particular contexts. I do not want to evaluate the lessons, but rather point to some events and actions which fostered or hindered pupils' learning (as far as I can say that on the basis of evidence I have available).

The evidence I have at my disposal is the video-recording of the lesson, the teacher's reflection of the lesson written when he saw it on the video and the national coordinator's and researcher's comments of the lesson. Unfortunately, the pupils' voices are not heard as there were no interviews made with them, neither were they asked to reflect on the lesson.

Even with the teacher's written reflection we cannot be sure that we interpret his/her action as it was meant, as the reflection does not include all the influences which were at play (the teacher either does not want to reveal them, does not consider them important or is not even aware of them). Baird (2001) presents a taxonomy of influence, with each level influencing the level below:

- Teacher *Values/Beliefs*
- Teacher *Intentions/Purposes*
- *Approaches/ Behaviours* exhibited by teacher
- *Methods/Procedures* selected by teacher to foster pupil learning

Only the third and fourth types of influence can be observed to a certain extent on the video. The teacher's intentions and purposes can perhaps be seen in his/her written reflection and only marginally we can infer about his/her values and beliefs.

Finally, we also have to be aware of a potentially negative influence of the video camera on both the teacher and the pupils. Pupils are often reluctant to speak in front of the video camera and may become more passive than usual.

LESSON A: PERIMETER OF CIRCLE

The first illustration concerns a Grade 8 lesson in which there were 15 pupils. The lesson started with the revision of the perimeter of a square, rectangle and triangle, which the pupils already knew. The teacher drew a circle on the blackboard.

Consider the following extract from the lesson. T stands for teacher, P stands for a single pupil, PP stands for several pupils, dots mean that there was a pause.

T Perimeter of a circle, circumference. How to go about solving it, that is a question. ... People were solving these problems as early as 2,000 years ago when Mr. Archimedes figured it out. He was able to do it. To find the perimeter of a circle. Do you have an idea? ... John?

P I would make a square there.

T Where?

P May I come up to the board?

T You may. [The pupil is drawing a square on the blackboard.] ... Good. I know the perimeter of a square. I can figure that out easily. Moreover, I have to realise the properties of the circle, that what John has drawn here corresponds to what? ... What does this correspond to?

P Radius.

T No, no. To what?

P Secant of circumference.

T That is correct as well. What is this? [Shows the diameter of the square.]

PP Radius.

T Excuse me?

P Length of the square.

T Yes, so this will be the length of the square. [Marks the side of the square *a*.] ... And in the circle, what does the blue segment mean?

P Straight line.

T It's not a straight line, it's a segment.

P (inaudible)

T Diameter. Good, yes, diameter. ... So, John suggests here the diameter of the circle, to compare it with the side of the square. [Writes $d = a$.] And from that he could figure out the perimeter of a circle. ... Yes? ... What did he do? He substituted the circumference with the perimeter of a circle. [He meant square.] Is that all right? Michael?

P It's not accurate.

T It's not accurate. Simply, it's not accurate. The shape is clearly different. I just can't replace it so simply. So, I'll do it more accurately. Which more accurate method could I use?

This is what the teacher says about this part of the lesson in his reflection:

> A1: I'm trying to create pupils' interest by assigning a problem for which the solution is not clear at first. In order to solve it, pupils must use their knowledge of the perimeter of a square and hexagon. When explaining, I'm also reviewing properties of the diameter and radius of a circle. To increase motivation, I include a historical note about Archimedes.

Pr2. Teacher presents pupils with challenging problems

Good tasks should be sufficiently easy for pupils to be able to use their previous mathematical knowledge, adequately complex to get joy from a mathematical discovery and sufficiently open to provide space for pupils' strategies. In Dewey's words:

> A large part of the art of instruction lies in making the difficulty of new problems large enough to challenge thought, and small enough so that, in addition to the confusion naturally attending the novel elements, there shall be luminous familiar spots from which helpful suggestions may spring. (Dewey, 1966)

We will distinguish between a learning task and a practice task. A *learning task* aims at teaching pupils something new and "the sequence of learning tasks show a coherent development of the object of learning" (Mok & Kaur, 2006, p. 148). A *practice task* mostly requires the repetition of a taught skill. Whether the task is a learning task (or a problem) or a practice one (or exercise) depends on (a) the way it is used in the classroom and (b) on the solver. For one pupil a task may be a learning situation, for another it may be a practice task.

Tasks can have many different forms dependent on the topic and the aim we have when using them in the class. But by themselves they do not automatically ensure good results. This seems trivial but my experience with in-service and pre-service training courses shows that it is not so. It is necessary to distinguish between the *potential of the problem* and its *realisation* and thus it seems best to analyse the tasks actually used in the classroom. A similar idea is expressed by Mok and Kaur (2006) who propose to investigate "learning task lesson event", that is the realisation of the task in the lesson.

The actual use of tasks in mathematics lessons in seven countries was also analysed during TIMSS 1999 Video Study (Hiebert et al., 2003). Research teams analysed about 100 lessons from each country. One aspect they followed was the percentage of so called 'Using procedures tasks' (solved by applying procedures), 'Stating concepts tasks' (tasks which call for a mathematical convention or an example of a mathematical concept) and 'Making connections tasks' (focusing on constructing relationships among mathematical ideas, facts, or procedures). Moreover, they also studied whether each type of task was used 'appropriately' in the classroom itself and found out that the potential of many 'Making connections tasks" had not been realised at all.

A potentially ‘good’ problem can be ruined if the teacher gives pupils a series of hints leading towards the solution step by step, reveals a solution prematurely to them, points to mistakes without giving the pupils time to find them first, leads them towards the strategy he/she considers the best, etc.

Comments: The situation in Lesson A was quite positive for pupils’ mathematical work. They felt motivated, the problem caught their attention and seemed to be challenging for them. They were able to suggest a way to start solution and to formulate that the first approximation was too gross.

The lesson continued and after about 2 minutes another fruitful suggestion came.

1	P	Could I draw the square in the inside?
2	T	Yes, that would work. That’s a good idea. Draw it inside. Inscribe it. The difficulty is that when I inscribe the square, however simple it is here, one side of the square equals the diameter, if I inscribe it, when... should I do it? I don’t want to. Do you know why? Because the picture... (inaudible). You’re correct. But the diameter is then what? ... The diameter in that square is what? ... The diameter is in the square? ...
3	P	Diagonal.
4	T	Diagonal. The diameter in the square is a diagonal. [...] The perimeter of the square would be a problem in this case. Why? Because you don’t know for the square ...
5	P	The lengths.
6	T	Sides. Now you could calculate it in fact, since you know the square root and from that you would be able to calculate the area of the square and from that you could calculate the sides. But you would not be able to figure out the side simply. So you cannot determine the perimeter of the square. Another way.
7	PP	So, I would for example use an octagon. ... Yes, octagon.
8	T	Octagon. Again, you will not be able to determine the sides. But where can you determine the side easily?
9	P	Triangle?
10	T	No. Not the triangle. ... Stanley?
11	P	Hexagon.
12	T	Hexagon. Do you remember? You take a radius, mark it here and here. [Shows on the blackboard with a pair of compasses how to construct a hexagon.] The radius. And what comes out of that ... is what?
13	P	Hexagon.
14	T	Hexagon. What is the length of its side?
15	P	Radius, radius of the circle.
16	T	Yes, that is how I made it, with the radius. I measured it with the radius. So the length of the side of the hexagon is the length of the radius.
17	P	It still isn’t accurate.
18	T	It still isn’t accurate. So how would I make it more precise?
19	PP	16-gon. 16-gon. 12-gon.

20	T	Wait. 16-gon?
21	PP	12-gon, 8-gon... 12-gon ...
22	T	Which one?
23	P	12-gon.
24	T	12-gon, yes. That means I would mark points here. Next. You wouldn't be happy again ... this will be a 12-gon. ... Then you can do ... [Some pupil suggests something and the teacher shakes his head disapprovingly and continues.] 24-gon. Forty...? [Pupils are saying aloud something inaudible but the teacher does not react.] Up to 96-gon. You would not be able to do it anyway, you couldn't do even the 12-gon.
25	P	But the more angles... [Teacher interrupts and finishes the idea himself.]
26	T	But the more angles, the closer it is to the circle and then I can assign the perimeter of the polygon nearly very precisely to ... to the perimeter of the circle. That's the procedure Mr. Archimedes used. ... Yes? We can write a note. [Writes on the blackboard.] Archimedes used for his calculation the 96-gon. The 96-gon. ... You can read it in your textbook that this was in the year 300 B.C. [...] He did this calculation. I will get back to that later but from these ideas which we made with the help of the square and hexagon, we are coming to one thing. ... That the perimeter of a square is four diameters [Writes it on the blackboard symbolically.] ... the perimeter of a hexagon ... David?
27	P	... Six.
28	T	Six of what?
29	PP	Diameters. [The teacher shakes his head.] Radii.
30	T	Radii. Six radiii.

Then the teacher showed on the blackboard that the perimeter of the circle lay somewhere between three times diameter and four times diameter. He stated that this was still very imprecise but it would suffice for their purposes. He wrote the result on the blackboard: The length of the circle depends on ... and the pupils substituted radius and diameter. After he pointed to the relationship of this deduction and direct proportionality which the pupils had done earlier, he stated that the number they were looking for was 3.14 or imprecisely 22/7.

This is what the teacher wrote about this part of the lesson in his reflection:

> A2: During the teacher-pupil dialogue, the pupils are offering possible solutions which are, however, limited by their abilities. Complicated thinking and deriving is a basis for a final solution that will be found in the form of a rounded number. Pupils are experiencing the same thinking patterns as those used in history.

Pr3. The teacher makes every effort to provide pupils with active learning situations

What do active learning situations mean? It means that pupils are put into situations which motivate them to actively participate in learning. It does not mean a kind of pseudo-activity when pupils solve mechanically exercises or when they find themselves the target of the teacher's partial questions without actually being aware of what they are supposed to achieve. Mathematics is understood as an action, a pupil's action. Baird (2001) proposes four types of active learning. The first being "receptive, compliant" in which the pupil only answers questions. For him/her, being active means being conscientious and busily engaged. The next two types are "curious, task-centred". In these the pupil either asks questions to find out knowledge required to complete a task component (type 2) or asks questions centred upon the completion of the whole task (questions are broadly strategic and managerial) (type 3). The last, fourth type is "curious, learning-centred" when the pupil asks questions that evaluate the nature, purpose and progress of learning. The higher the type, the higher the level of pupils' reflection and personal responsibility and control over personal practice. I understand 'asking questions' in the characterisations of the above types as either asking questions aloud or in one's mind. Therefore, it is not easy to describe how active the pupil is on the basis of a video-recording only.

It is obvious that the phrase 'makes every effort' in the name of this principle is very important. It is unrealistic to suppose that all the pupils will be motivated to their own activity even by the best of teachers.

Comments: In the presented episode we could see that the teacher really strives to provide pupils with active learning environments. He asks questions, he wants the pupils to be active and this effort is successful at the beginning. Pupils answer questions (e.g., 3, 5, 7, etc.) and sometimes even supply their own ideas (e.g., 1, 7, 17, 25). Unfortunately, these instances are not encouraged by the teacher, he sometimes ignores them (e.g., 24) and continues according to his preliminary agenda (square – hexagon – 12-gon – 24-gon – 48-gon – 96-gon).

Pr4. The teacher develops pupils' ability to think independently and critically

By that I mean that the teacher should promote an atmosphere which supports enquiry.

> Enquiry depends upon asking questions, constructing argument and critically reflecting on outcomes. (Burton, 2004)

Independent does not mean that pupils should work on their own. Social interaction is important too (see principle 6). Independency relates rather to the need to be less dependent on the teacher.

As I see it there are (at least) four necessary prerequisites for this principle to be enacted in the classroom.

1. It is necessary to overcome an idea that mathematics could be presented to pupils in a ready-made structure so that they are not delayed by dead ends which the teacher knows lead nowhere (Kubínová, 2002) and to see these *dead ends* as necessary building blocks of the world of mathematics which is being built in the pupil's mind.

2. Pupils must be given sufficient *time to think*. Often, when observing the teaching practice of our students, I see them ask good questions but they rush through the answers in such a way that even the most able of pupils do not have time to really think it over.
3. The next prerequisite is best formulated by Glasersfeld (1997, p. 181): "Teachers should never fail to manifest the belief that students are capable of thinking."
4. It is necessary to encourage pupils to ask questions. In practice it is very often the case that pupils are flooded with questions they did not ask and their own questions are left unanswered (Kurina, 2002b).

The teacher feels a natural inclination to help pupils when they are stuck. However, there is a delicate boundary between *guiding and funnelling*. Bauersfeld (1988) introduces the term funnelling for a certain kind of communication pattern in which the teacher has an idea where he/she wants to take the pupils and leads them there by asking gradually more and more narrow questions. Pupils often have only the opportunity to say yes or no or to answer by very short sentences or by one word. They often cannot guess where the teacher is heading and the teacher more often than not answers his/her questions him/herself.

> Thus, guessing their way through implies the possibility that the students will concentrate on guessing rather than on the mathematical content of the subject. (Alro & Skovsmose, 2004)

Comments: What happened in the presented lesson in terms of this principle? Does the teacher <u>at this particular time</u> fully use the potential of the situation to develop pupils' thinking? I think not. Some actions and utterances could inhibit mathematical thinking. For example, utterance number 4 or 8 in which he refuses pupils' suggestions by saying that they will not be able to calculate. It could be true, but in my opinion, the teacher could have given a justification why the calculation would be too difficult or even let them try it first. I also think that pupils did not have sufficient time to think about the teacher's questions. They were expected to imagine the situation in their heads as they did not have time to make any drawings and think the problem over before responding to prompts.

As I see it, the teacher's reaction to the pupils' suggestion of octagon (see 8) is one of the key parts of the lesson. The pupils probably considered an octagon as a logical continuation of the square. The teacher's poorly justified and for pupils most probably incomprehensible refusal of the octagon (see 8) and afterwards of the triangle (see 10) and 16-gon (see 20, a logical continuation of the octagon for pupils) is likely to create the impression that they were playing a game of 'guess what I am thinking' (e.g. Young, 1992, Alro & Skovsmose, 2004). From the video-recording it is clear that at least some pupils were able to grasp the limiting process, however, the teacher's reactions might lead them to believing that in this process some n-gons *must be omitted. Let us add, that from the point of view of the teacher, the refusal of the octagon was perfectly logical – he was aiming at the 96-gon used by Archimedes by doubling the number of sides beginning with a hexagon.*

Finally, I do not think that the presented extract is an example of clear funnelling as from the reactions we can infer that at least some of the pupils were aware of where the teacher is heading.

Summary for Lesson A

The teacher's intentions are good; he wants to show pupils a nice deduction of the formula for the perimeter of the circle. He had an agenda which was actually plausible but it was enacted in an inappropriate way. He was unable to change it according to the pupils' suggestions. In the sense of teacher's flexibility (Leikin & Dinur, 2003: "Teachers' flexibility is expressed in their adaptation of the initial agenda to [pupils' real] needs and processes."), the teacher was, at this moment, inflexible. If we look in his reflection, even in his comment to the first part of the lesson, he says that pupils will need the perimeter of a square and a hexagon. He did not take into account other possible suggestions.

In terms of the other principles I cannot say much. There was no communication between pupils, and one-way communication with the teacher. The problem was challenging for at least some of the pupils particularly at the beginning of the lesson. It stopped being challenging for those who started playing the game 'guess what the teacher thinks'. But there may be pupils who followed the teacher and were on the right track. The video-recording does not provide us with sufficient detail.

To sum up, if the teacher normally follows this pattern of teaching, he develops an image of mathematics being done only by those who are mathematically more gifted. Pupils have to learn the results and are not able to discover anything by themselves (see also the teacher's reflection below).

The lesson is always embedded in context so we do not actually know whether this is a normal way of teaching for this particular teacher. [5] But his written reflection says a lot. In it, he seems to think that he led the pupils to discover the formula for themselves and does not see (or admits to the reader) any deficiencies in his reactions. Another teacher in a similar situation wrote in her written reflection of her lesson the following (the lesson is also from TIMSS 1999 Video Study):

> I wish I would have let the pupils have more control here. I was very much guiding the lesson. I was leading to the correct answers. [...] I don't think that this was the best response I could have given this pupil. I should have praised her more for finding another method and added it to our other information, rather than possibly making her feel like her method was too lengthy.

It is interesting that when I use this lesson with pre-service and in-service teachers, they are at first positive about the part presented here. They value the teacher's intention that he wants to introduce pupils into a mathematical way of thinking. For example, they write: “The pupils participated in the discovery of relationships, they were not afraid to express their opinion, the teacher did not refute any proposed hypothesis and he tried to make use of it.” Only when we do a more detailed analysis, when they are asked to pay attention mainly to the teacher’s reaction to pupils’ suggestions, do they begin to realise that the teacher indeed refuted some suggestions and to discuss its possible implications. I presume that in their first seeing the lesson, they devote more effort to understanding the mathematical basis of the lesson (where the teacher is heading) and only afterwards they are able to consider the pupils’ side as well.

LESSON B: AREA OF TRIANGLE

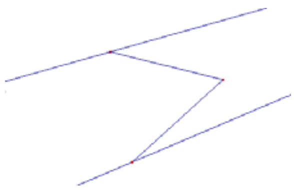

Fig. 2

Lesson B, again Grade 8, with 35 pupils in the class. The teacher revised the knowledge from the last lesson on a TV screen: if the triangles have the same base and the same height, their areas are the same (the triangles were drawn between two parallel lines). Then he drew a picture on the board (Fig. 2) and told the pupils a story.

T There is Jack’s land. Okay? Over here is Peter’s land, okay? [Laughter.] [...] And these two people’s, uhm, borderline is bent like this but we want to make it straight, okay? [...] Jack, is it okay with it around here?

P Yes.

T Is that okay? [Laughter.] Then, we’ll end today’s class, okay? [Laughter.] Peter, is it okay with it around here?

P Ahhh.

T No?

P No.

T Around where would you like it?

P It would be better if mine was wider.

PP A lot more to the bottom?

T Huh? A little more over here?

PP More. [...]

The conversation continued for some time in a similar way and the teacher moved his fescue according to the pupils’ suggestions so that they saw that some divisions would be unjust to one of the land owners. The pupils were relaxed and there was a lot of laughing. Then the teacher asked the pupils to estimate where the borderline might be and asked one pupil to come to the board and present her estimation. The girl showed a line going through the middle of the two segments.

Then the teacher repeated the task. The pupils should make the borderline straight so that the areas of both the plots stayed the same. The pupils were asked first to think about the task individually for three minutes and then they could work in pairs if they wanted to, they could discuss things with the assistant teacher or they could consult a hint card. While the class was working, the teacher moved around the classroom and encouraged pupils, never actually giving other hints than "draw the figure", "remember what we did last time", etc. The activity took about 15 minutes and then two pupils presented their solution in front of the class (a re-drawing of the solution is in Fig. 3).

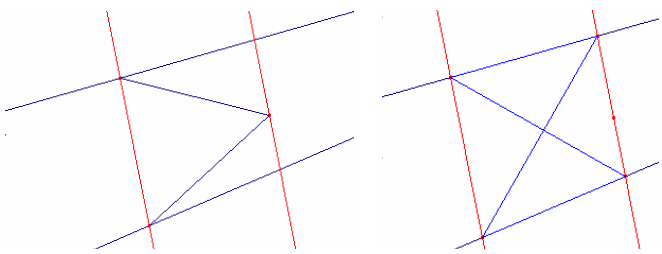

Fig. 3

Note: Unlike lesson A, lesson B was not a teacher-pupils dialogue so it is quite difficult to present parts of the lesson in extracts of dialogues. The teacher worked with individual pupils and commented on their solution rather than drawing everything on the blackboard.

Pr1. The teacher arouses the pupil's interest in and joy from mathematics

By this I do not only mean motivation by real (or rather pseudo-real) problems, but by the mathematics itself. Ideally, pupils should be curious to see what the result is and what processes lead to it. The task should resonate with the pupils' own experience – from real life and/or mathematics. Sometimes a purely imaginary context can be motivational. Many studies have shown that intrinsic motivation can be sustained if the mathematics learning brings about some positive emotional experience. That is the teacher should enable each pupil who makes an effort to experience joy from a mathematical discovery, however, trivial.

> Experiencing 'small intuitive breakthroughs' is dependent, then, on students being put in situations where they experience and reflect upon the power of such breakthroughs, in order to learn how to access that power. This cannot happen in a classroom that is 'delivering' facts or tools where the purpose is defined by being a required part of a syllabus. The attention and commitment of the students to the activity on which they are engaged is a necessary part of ensuring that they experiment with recognizing and using their intuitions. ... All of this requires that the students are actively engaged with mathematical problem solving and not with repetitious or rote-learning practices. (Burton, 2004, p. 186)

A mathematical discovery is very powerful and provides pupils with long lasting impression. Consider the following statement of a 13-year-old pupil (it comes from J. Hanusová's work):

> At the beginning of the lesson I thought that I would fool around, that I would not be interested in the subject matter, but after we had discovered correct answers to the first

> task, I suddenly started to like it. We were happy when we discovered something new. I felt a strange pulsation somewhere inside me. A feeling that I understand something and furthermore, it interests me.

In the following situation the teacher's action does not arouse pupils' interest in mathematics.

> The pupils got a worksheet with some mathematical problems which were accompanied by pictures. By solving problems, they should find out the colour they should use for the pictures. Five minutes before the break, the teacher said: "In the rest of the lesson you can colour the pictures, but silently. If you are not silent, we will calculate."

The teacher wanted to motivate pupils towards mathematics by using pictures with the problems but by her commentary she achieved completely the opposite. Mathematics is only used as punishment, she suggests that she considers colouring to be more enjoyable than calculating.

Comments: The task used in Lesson B is a real problem for pupils even though, in fact, it is used as practice of what was done the previous lesson. The motivating way the teacher presents it using the pupils themselves sets the scene in a pleasant way and as far as we can infer from the video, the pupils work with enthusiasm. The realness of the problem also supports the teacher's claim that the solution must be precise, mathematical. For example, when one pupil showed an approximate solution, the teacher said: "Wouldn't they fight over it if it was approximate?"

Pr6. The teacher initiates and moderates discussions among pupils on the mathematical basis of problems [6]

I included the mathematical basis of problems in the title because the challenge to support discussion can sometimes lead to 'empty' discussions in which pupils quickly learn to make unsupported claims in order to delay teaching. In this respect, we can speak about productive dialogues (David & Lopes, 2002), that is "dialogues in which the students' talk shows the use of elements of mathematical thinking". [7]

Rittenhouse (1998, cited in David & Lopes, 2002) describes two roles in which a teacher finds him/herself during discussions with pupils:

> Stepping In – when the teacher participates in the discussion with the students, listening to their ideas, asking questions and contributing to the discussion by providing insights according to her own mathematical knowledge.
>
> Stepping Out – when the teacher offers commentary and formally teaches the rules and norms of the discourse he/she wished her students to use as they talked about mathematics.

Similarly David and Lopes (2002) speak about the teacher being a participant versus commentator. Instances in which pupils develop mathematical knowledge in a productive dialogue with only prompts from the teacher and with the teacher being a commentator are, in my opinion, quite rare in Czech primary schools.

When the task is set, the teacher's task is to monitor and/or guide pupils. *Monitoring* is the process by which "the teacher observes the progress of on-task activities and homework, ascertains student understanding, or selects student work, with intent to keep track of student progress, question student comprehension and record student achievement" (O'Keefe, Hua Xu & Clarke, 2006). *Guiding* is the process by which "the teacher gives information, elicits student response in order to promote reflection, or facilitates engagement in classroom activity, with intent to actively scaffold the development of student participation and comprehension of subject matter" (ibid).

Comments: Let us consider lesson B from the point of view of this principle. The teacher's management of learning consists of three stages. The individual work enables the pupils to realise what the problem is and what knowledge they might need. Their attention is focused on the problem. They are not disturbed by anyone at this time.

During the next stage of pair work (or group work), pupils make their ideas explicit to each other and discuss them. They move freely around the classroom. They mostly work in pairs. Unfortunately, we cannot listen to their discussions as the microphone is focused on the teacher. From his utterances and pupils' responses, we can infer that he is both monitoring them (asking what they are doing) and guiding if lost. This is done in the following pattern: First, the pupils are asked to draw a triangle by completing the two line segments with the third. Then they are asked to tilt their heads so that they see the triangle in their picture in the same position as the ones on the TV screen (they are in a horizontal position there) so that they can see that their previous knowledge can be used for solution. Most pupils are then able to border the triangle with two horizontal lines (see Fig. 3).

In the last stage of the work, there is a public presentation of pupils' solutions done by pupils and this is followed by the teacher's summary. The teacher lets two pupils present their solutions even though they are not able to do so very precisely.

Pr5. The teacher understands a mistake as a developmental stage of the pupil's understanding of mathematics and an impulse for further work

> To improve a pupil's mathematical self-confidence, the teacher's praise and good marks are not enough. It is necessary to get the inner feeling of joy from overcoming an obstacle. This feeling can also be brought about by a mistaken idea. The joy does not come from the correctness of the result only but from the effort used to achieve the result. Even the teacher who finally has to lead the pupil towards the idea that the result is incorrect, can keep the idea in the pupil's mind that intellectual work can be a source of a great joy. (Hejny, 2004, p. 74)

Naturally, the teacher who believes that what counts is the product, the result, has problems with praising a pupil's reasoning which has led to the incorrect result.

In schools we often see a completely negative attitude to the mistake (neither pupil, nor teacher should make a mistake). But even if the teacher does not have such a view, he/she often corrects the mistake without any effort to find the reason for it or without encouraging a pupil to find the mistake him/herself. Pupils quickly learn that the teacher is the authority and decide~~s~~ whether the result is correct or not. Consider the following statement from a teacher (Cachová, 2003):

> When I ask pupils what the absolute value of a real number is, I mostly learn that it is a positive number. So the other day I told the pupil that the absolute value of –5 is 15?! Number 15 is positive so it follows your definition. For a time he looked at me strangely and so I said: Disprove it! And he says if you are saying so, it will probably be true... The class burst out laughing, so did I, but one should rather cry and think.

Comments: If we consider Lesson B from the point of view of this principle, we can find, for example, the following instances in which the teacher expresses the idea that he is there because of the child and his/her learning:

T No... well, okay? That is okay? The mistake is what's important. [...]

T Because okay? If people can do it from the beginning then they don't have to come to school. [...]

T It's not okay. What? Don't worry about me. It's you who is learning. [...]

T But the way of thinking was good. It was quite a sharp thought, okay? The assumption was good.

TEACHER A' AND TEACHER B'S POSSIBLE VALUES

The two illustrations presented here are quite different. Their apparent goal is similar: to let pupils experience thinking in a mathematical way. Their methods of doing this is quite different though. I would like to emphasise here the apparently different focii of the two teachers.

Let us consider not only what can be seen on the video but also what the teachers wrote in their reflections of the lesson.

Parts of reflection of Teacher A

Only A1–A7 concern the parts of the lesson presented in this text (A1 and A2 were given above).

> A3: I consider it very important that the pupils write down the fact that the author of this procedure is Archimedes. It is a way to honour an important personality.

> A4: I usually write and draw important information (conclusions, formulas, diagrams, pictures) on the board, and I expect the pupils to have these in their notebooks. Most of the time they carefully copy everything that's on the board. I believe that by copying what's on the board pupils are already learning and developing new knowledge.

> A5: The pupils are taught to start the thinking process with assumptions and finish it with conclusions.

A6: I consider the aesthetics of mathematical thinking and writing to be a very important feature.

A7: The value of a number Pi can be found in the mathematical tables. Pupils also don't have to remember the formulas, they will find them in mathematical tables as well. With the frequent use of the formulas many pupils will eventually remember them.

A8: In most cases, after the explanation of a new topic, I usually assign example problems.

A9: I usually solve an 'example problem' on the board on my own, so pupils have the opportunity to see and follow the logistical and formal appearance of the solution.

A10: In today's class, the assignments are so simple that pupils can do it on their own. The problem requires only substitution of the numerical values into the formula (diameter or radius), and the calculation of the circumference.

A11: During the practice, I call up pupils to the board. I check and correct their mistakes. Some pupils can recognize the mistakes. I try to assess pupils' work and make some comments (e.g. compliment, point out problems).

A12: Homework is selected from the textbook. I expect pupils to complete the work the same way as it was done in class.

A13: In this particular class, pupils worked well and they were active, but I did not praise their work too much.

A14: I think I should have praised the pupils' work more during this lesson and reviewed some of the key moments of the lesson.

If we put these reflections together with what can be seen on the video, we can make some more or less accurate deductions of teacher A's values.

For teacher A, the mathematical aspect is very important, he is excited about mathematics and he wants the same for his pupils (A1, A3, A5, A6). He tries to show them how the knowledge has been developed in history and he believes that he actually built a staircase for them which will lead them towards the abstract knowledge, the formula for the circumference of a circle (A2). But did he succeed in this particular case? The formula was deduced and by the end of the lesson practiced, too. Was there any difference between the learning which the pupils would have made if the formula had been just given to them as a ready-made product? For some pupils yes. Who can say whether the boy sitting at the back of the classroom who was active at the beginning of the lesson grasped the idea of gradual improvement of the solution? But for most, I would say that the 'limiting process' was not well grasped. Thanks to the teacher, the pupils probably thought that they could not choose whatever polygon they wanted (with gradually more sides) but that for some mysterious reasons, they have to skip some (a hexagon is useful but an octagon is not).

What else can we say about the teacher's style and beliefs? Mathematics should be presented in a systematic way, by explanations followed by practice which brings about remembering (A4, A5, A7). Pupils learn best by imitating what is shown by the teacher; they are supposed to use the same procedures as the teacher both in class and at home (A4, A9, A11, A12). On the other hand, mathematics is not about remembering formulas (A7). Even though he emphasises the aesthetics of mathematical thinking, there is no mentioning in his reflection that learning mathematics is connected to pupils' thinking or that different solving strategies might appear. Mistakes are not welcome and must be corrected by teachers (A11, A12). The correctness of what pupils say is evaluated by the teachers (A11) and the teacher should praise them if they work hard and correctly (A13, A14). By themselves, pupils can only solve very easy tasks (A10). Overall, pupils are totally dependent on the teacher.

For teacher A, mathematics is central to his thinking and his role is to show pupils what he considers to be the best strategies. The pupils' role is to imitate them.

Parts of reflection of Teacher B

Only B1–B9 concern the parts of the lesson presented in this text.

B1: Review of the last class: This is done to help accomplish the goal of the current lesson, but I make a point of not taking too much time. To have the pupils visually grasp the many possibilities, I explain using a computer with a moving image on the screen.

B2: I have the pupils keep their textbooks closed to make them think independently.

B3: To arouse the pupils' interest and connection to the problem, I set up the opening of the lesson by using two pupils' names. This also makes the class atmosphere cheerful.

B4: I have the pupils tackle the problem with focus by limiting the time they are allotted to think about it.

B5: With two teachers we support and assist the pupils who are not able to come up with a solution idea: We have them recollect the previous lesson. We encourage and praise them.

B6: We acknowledge the different methods each person thought of using and praise each other for solving the problem and working hard on it.

B7: I support and assist the pupils to come up with their own ideas.

B8: I want the pupils to find more than one answer, as many different solution methods as possible. Furthermore, I want the able pupils to think about how many ways there actually are to solve the problem.

B9: I have the pupils raise their hands to confirm the methods they used and give them a sense of satisfaction and accomplishment.

B10: I pose a developmental problem to the pupils who show interest. I assign it as homework and make it open-ended.

> B11: I praise the pupils for participating enthusiastically in the problem solving during class.

For this teacher, the pupils are at the foreground (B3, B5, B6, B7, B8, B9, B11) while not losing sight of the mathematics he wants them to acquire. We can see constant references to pupils and their understanding, their activity, their independence. In mathematics, often several routes exist thus it is valuable when the pupils find different strategies (B6, B8). The teacher's role is to support pupils in their learning (B5, B7). Pupils' ideas are valuable and their feelings concerning mathematics matter (B3, B5, B6, B9). They are capable of finding their own solutions; they can solve open problems, too, without the teacher's help (B7, B10). Overall, I would say that pupils are led to develop an internal sense of authority.

For teacher B, children are central to his thinking and his role is to support their learning. He uses their ideas and realises their potential for solving problems on their own. He gives his pupils confidence.

The presented method of putting together the impressions from watching the video and what the teacher actually wrote in his/her reflection of the lesson is a good activity for student-teachers and in-service teachers. They try to find characteristics of the teacher's teaching style and his/her values (the intended as well as implemented ones, Bishop, Seah & Chi, 2003) in the two types of data. While doing so they contrast these with their own values and preferences and become more aware of some phenomena not visible at first sight. This is done in a practical and accessible way. Actually we can see a parallel with mathematics learning here. The teachers and student-teachers construct these phenomena themselves, they are not given by me or by a scientific article.

Of course, again, while doing the above we must be aware of the incompleteness and inaccuracy of our deductions about the teacher considering the lack of information available.

PR7. TEACHER EMPHASISES THE DIAGNOSIS OF THE PUPIL'S UNDERSTANDING RATHER THAN THE REPRODUCTION OF ANSWERS

This principle cannot be easily illustrated by the two lessons above, so for the sake of completeness I will include a short illustration here.

Mathematics lessons often lead pupils to be able to react to questions quickly and without mistakes. The questions are often yes-no questions or very close ones and sometimes even include a part of the answer. This, however, does not diagnose a pupil's understanding. This understanding is better diagnosed by a non-standard problem.

> The pupil is being examined on Thales' Theorem. He is not able to express his idea without drawing and thus desperately draws the whole situation with his hand in the air and slowly begins to put the theorem together. The teacher evaluates his performance by

the worst mark saying: "You must have it at your fingertips!" (extract from a Czech lesson)

We can suppose that the pupil would be able to explain the theorem if he was allowed to draw. However, the teacher values more when he is able to say it quickly and without hesitation. The signal the pupil might have received is "it is not important if you understand it, the main thing is to be able to say it".

PRINCIPLES IN MY PRACTICE [8]

I have implemented the above principles in my university teaching, mainly in *the course of 'Analytic Geometry' for future mathematics teaching*. The traditional form of teaching in which students are presented with ready-made products in the form 'definition – theorem – proof' has been violated in this course in that the students are asked to build mathematics for themselves. They are given problems and challenges and they have to devise a way of tackling them, to discover their own solving strategies, to formulate hypotheses and test them (Stehlíková, 2002).

Based on my research and my teaching, I will make some tentative conclusions concerning the above principles-driven teaching which I advocate here.

Students get to understand that mathematics is not only a set of organised facts which they learn about but also an unknown area which they can explore. It is our experience that because our students believed that they were discovering something really new which cannot be found in textbooks, they felt strongly motivated (see my research on structuring mathematical knowledge, Stehlíková, 2004). Teachers emphasising procedures do not enable students to view mathematics as an exploratory science (Cooney, 2001).

It is necessary to overcome the students' expectations based on their previous experience that (a) the teacher will give them many hints, or even instructions, and help them a lot, and (b) that any mathematical problem can be solved by applying known algorithms. Moreover, it is necessary that students experience success quite early in their work and that they feel it as a success. So suitable problems must be looked for. [9]

The teacher's approach is important. Students' success must be felt as positive by the teacher. There is no competition between students and teacher, it is a collaborative process in which the participants have the same goal but different responsibilities. The students should be encouraged to rely more on themselves and to develop an internal sense of authority (confidence).

We hope that exposing students to teaching based on the seven principles above will eventually contribute to their teaching in a similar way. However, their didactic experience might be firmly rooted in the context of the subject/course or in the level of mathematics which was taught in this way. In my research (Stehlíková, 2004) one of the students, Molly, valued her independent discovery of some parts of mathematics, however, on the other hand, she also appreciated those courses in which the teachers "proceeded systematically and explained things clearly. All parts are logically ordered.". She understood the relationship between the mathematics introduced to her in an exploratory way and the traditional mathematics taught in her university courses which was mostly in a transmission way, as complementary not antagonistic. In the final interview, she appreciated traditional courses for their understandability and clarity, and the exploratory approach for bringing her joy and self-satisfaction. She also stressed that only some parts of mathematics should be presented in an exploratory way and only for older students: "... at the secondary school, it is too early, they do not have the basic knowledge." Her opinions are mostly based on her own experience. As found out in the final interview, she had never had an opportunity to experience other than transmission teaching at her secondary and primary schools.

Even in mathematically productive dialogues, I believe, students should be made aware of the mathematical basis, for example, by the teacher's summary (the teacher being in the role of a commentator). In the course on analytical geometry, I presumed that the students would be able to gradually build in their head a structure of what we were learning. However, I found out that often they needed me to make them aware of the underlying mathematics. It often happened that the students' awareness of the structure of the mathematics knowledge they were learning came about only during the discussion with the teacher during their oral examination. Similarly, Jirotková in her university course for future elementary teachers, which is run in a dialogic way, reached a similar conclusion. Many students get an idea that there was nothing for them to learn as they "do not have to learn any formulas by heart, no definitions, no theorem and no proofs. Students, who were previously led to 'acquiring' knowledge by rote learning, do not value the importance of thinking, searching and critical appraisal of concepts, relationships, situations." (Jirotková, 2004)

It is very difficult, if not impossible to reproduce the same didactic situation (see obsolescence of didactic situations, Brousseau, 1997) as "the practices observed are the products of multiple interactions whose elements are not always the same, even for any one teacher" (Bodin & Capponi, 1996). I can see this in my teaching each time I start working with a new group of students.

The common denominator of all the seven principles is that the teacher cannot plan his/her teaching in advance in all details. What he/she can do is to think over the problems and their solutions, and possible pupils' strategies and questions. Towers & Davis (2002) speak in this respect about the 'plan' being a kind of thought experiment or an exercise in anticipation. Here again the need to have a good knowledge of mathematics of the teacher stands out. Without that the teacher cannot guide the pupils on the path towards creating the world of mathematics in their head as he/she is not able to react adequately to pupils' suggestions which are not in line with his/her strategy. Moreover, if he/she tries to hide this deficiency behind direct teaching of facts and procedures, the damage on the pupils' perception of mathematics can be substantial.

The second way of using the above principles is in *the mathematics education seminar with future mathematics teachers* and in in-service courses as a basis of analysis of video and text cases which should promote teachers' reflection.

In accord with Tichá and Hospesová, I also found out that the joint reflection of students has rather anecdotal character at first with comments being restricted to the description of the teacher's action. They seem to "believe that a 'perfect' performance from the teacher is a prerequisite of 'good' learning. The performance and reactions of pupils, their cognitive development, was of less importance to them." (Tichá & Hospesová, 2006) The students as well as teachers who are exposed to the videos have to learn to watch them. Their attention must be focused on some aspects of the teaching process and the principles which I use have proved to be rather effective. They enable us to see the classroom through the peephole of one principle, to focus our attention on a certain characteristics.

Gradually, the students' attention changes from the general pedagogical climate (relaxed atmosphere, teacher being strict, etc.) to following the line of thought of the teacher (intended curriculum) versus what actually happened (implemented curriculum), of his/her possible values and how they might affect pupils.

Even though I have tried to point to deficiencies and dangers of the analysis of videos, I still believe that if we keep them in mind, the analysis of real teaching can be beneficial to in-service teachers and student teachers in particular. The analysis helps them become aware of some underlying characteristics of teaching in a practical way.

CONCLUSION

The principles presented in this article necessarily reflect my own beliefs and values and I believe that a reader may be able to infer many of them from what I have written above. Even though I have tried to express myself openly, there may be some of which I am not even aware myself.

Each 'ism' can be dangerous if one is trying to keep it too strictly. In the Czech schools, transmission teaching methods are far more frequent and thus, I advocate more *balance* so that the seven principles above are actually enacted in the classroom to the extent the situation allows.

When looking at some older citations, it appears that each new generation rediscovers anew what should be done in order to improve instruction. In a similar way I rediscovered what others before me formulated first and often in much better words. Consider what Eduard Čech, a prominent Czech mathematician, said in 1955.

> When teaching geometry (at the basic school), I think that it is necessary to follow four principles:

- Subject matter and its elaboration should provoke as big interest as possible. It is not a matter of teaching something but rather of achieving that children will look forward to lessons. It is necessary that children learn to love geometry.
- The teaching must be led in such a way that it provides many opportunities for pupils' own activity. The thirst for active work is something unstoppable at this age.
- The learning cannot have no concrete content. The pieces of knowledge must be ordered in such a way that they reappear again and again in later teaching.
- It is necessary that pupils meet something by examples which is not systematic yet but which gives a picture of how it will look later.

My cherished dream is that in my own teaching I would be able to inspire students towards mathematics and enable them to experience joy from mathematics in the same way as Milan Hejny did for me. Such a task is enormous and requires an everyday struggle.

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NOTES

1. It is not the aim of this article to enter into extensive discussions as to their meaning; the reader is referred to works by Glasersfeld (1997), Davis, Maher & Noddings (Eds.) (1990), and others.

2. For a survey of literature see Confrey & Kazak (2006).

3. When this article contains the perception of student-teachers, these are future mathematics teachers of pupils in the age-range 12-19 year old, and it comes from a year-long seminar for pre-service teachers (run twice so far).

4. Other authors describe the teacher's activity in the mathematics classroom, for example, Ponte (2001, p. 67) considers these roles of the teacher: 1. Challenge pupils. 2. Support pupils. 3. Evaluate pupils' progress. 4. Think mathematically. 5. Supply and recall information. 6. Promote pupils' reflection.

5. The teacher might have wanted to show a 'discovery lesson' for the sake of the video camera. He may not be used to this kind of work and that is why his reactions are not, in my opinion, always the best ones.

6. Much has been written on communication and interaction between the teacher and students (e.g., David & Lopes, 2002, Alro & Skovsmose, 2004).

7. From this point of view, the dialogue in Lesson A was productive; at least in some of its parts the pupils' reactions showed that they were thinking mathematically.

8. I have a problem finding a name for the kind of teaching I have tried to describe in this article. I do not want to use 'constructivist' or 'discovery' teaching for the connotations they evoke. Thus I use '7-principles-driven teaching'.

9. In the Czech research group, we have been looking for such contexts deliberately (e.g., Stehlíková, 2004 – a finite arithmetic structure, Zhouf, 2006 – arithmetic sequences of higher

orders, Hejny & Stehlikova, 2004 – trileg mini-geometry, Kratochvílová, 2001 – an infinite arithmetic structure).

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TEACHERS, TECHNOLOGIES AND THE STRUCTURES OF SCHOOLING

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Advocacy for new technologies is part of a wider reform pattern which has had limited success in changing well established structures of schooling. Contemporary theories of technological innovation and educational change acknowledge that these processes are shaped by the sense-making of the agents involved. Accordingly, a model of secondary mathematics teachers' ideals for classroom use of new technologies is presented, validated by several studies. But equally, these studies identify difficulties which teachers encounter in realising these ideals. They signal the importance of appropriately configuring key structuring features of classroom practice –working environment; resource system; activity format; curriculum script; time economy– and developing the craft knowledge of teachers correspondingly.

1 PERSPECTIVES ON TECHNOLOGY INTEGRATION

This talk will look at a particular challenge of innovation in the professional field of mathematics education. It is a challenge which has been highlighted by several of the thematic groups at this conference in their outlines of work and calls for papers: the challenge of renewing the practice of mathematics education in schools so as to take greater account of new technologies. The conference call from Working Group 9 on Tools and Technologies talks of how *"research in learning with technological tools has been showing much promise"*. It points specifically to the potential of *"learning environments where students have richer opportunities to construct mathematical meanings, to explore and experiment with mathematical ideas and to express these using a wealth of representations"*. However, the call notes that *"actual use of these tools in real school environments is still very thin despite the abundance of governmental funding"*. The call suggests that *"the changes in classroom practices involved in the use of the technology seem to pose a real challenge to administrators, curriculum designers, teachers and students"*. Accordingly the call seeks *"a deeper understanding of how the potential suggested by research… can be grounded… in classroom practices with respect to systemic schooling"*.

I want to throw light on this issue by adopting a rather different perspective. Contemporary studies of the social shaping of technology acknowledge that differing conceptions will arise of any technology; that there is always scope for what such studies term 'interpretative flexibility' (MacKenzie & Wacjman, 1999; Williams & Edge, 1996). These differing conceptions come into play not only during the evolution of the design of a technology, but in the course of its propagation as a finished product. They influence its non-adoption by potential users, or its appropriation by them. (Carroll, Howard, Vetere, Peck & Murphy, 2001; Rabardel & Waern, 2003). Equally, such conceptions and processes often lead to technologies being taken up in ways which may appear to their designers as something of a misappropriation. Accordingly, the concept of 'innofusion' has been developed in reaction against conventional technocratic models of design (Williams & Edge, 1996). It proposes that aspects of 'innovation' continue throughout the process of 'diffusion'. In this view, as a technology traverses a network of social actors, it is reframed in the light of their interests and circumstances (Tatnall & Gilding, 1999). In this interactive sociocultural model, design continues in usage (Rabardel & Bourmaud, 2003, p. 666).

Within education, the uptake and influence of new technologies has been limited. Reviewing the educational reception of wave upon wave of new technologies over a period of eighty years, Cuban (1986, 2001) suggests that a recurrent pattern of response can be found; a cycle in which initial exhilaration, then scientific credibility, give way to practical disappointment, and consequent recrimination. He reports that while new technologies have broadened teachers' instructional repertoires to a degree, they remain relatively marginal to classroom practice, and are rarely used for more than a fraction of the school week. As far as computers in school mathematics are concerned, a glance at the relevant statistics from the recent TIMSS studies will confirm this picture (Mullis et al., 2004).

For scholars of school reform, the reception of new technologies forms part of a much wider pattern of largely unsuccessful attempts to change the structures of curriculum, pedagogy and assessment at the heart of schooling. Cuban (1986: pp. 81-82) argues that the characteristic features of *"teacher-centred instruction"* evolved in the face of *"the implacable reality that policy makers institutionalised over a century ago"*. In this reality, *"a teacher is required to face thirty or more students in a classroom for a set period of time, maintain order, and inspire the class to learn content and skills mandated by the community"*. Cuban refers to this as *"the DNA of classroom life"*. It forms part of what Tyack & Tobin (1994: p. 454) term *"the grammar of schooling"*. These evocative metaphors seek to convey the way in which schooling has a core structure encoded in the three Cs of classroom, curriculum and certification. While each element is open to a degree of mutation and evolution, change is constrained by the system as a whole.

Critiquing this line of argument, Papert (1997) objects that it simply describes the *"defense mechanisms"* of schooling which serve to *"frustrate reform"* (p. 418). These mechanisms, Papert argues, *"are* concomitants *rather than* causes *of [its] stability"* (p. 419). What is agreed on both sides, however, is that while innovations set out to change schooling, a reciprocal process unfolds in which schooling changes innovations. This has also led Papert towards autocritique; towards a *"shift from a stance of reform to a stance of evolution"* (p. 418). He argues that *"the most insightful… teachers working in conventional schools understand what they are doing today… [as] not being the ideal they wish for"*. He suggests that *"as ideas multiply and as the ubiquitous computer presence solidifies, the prospects of deep* *change become more real"* (p. 423). In particular, he sees the day-to-day classroom work with computers of these teachers as the seeds from which such change will grow.

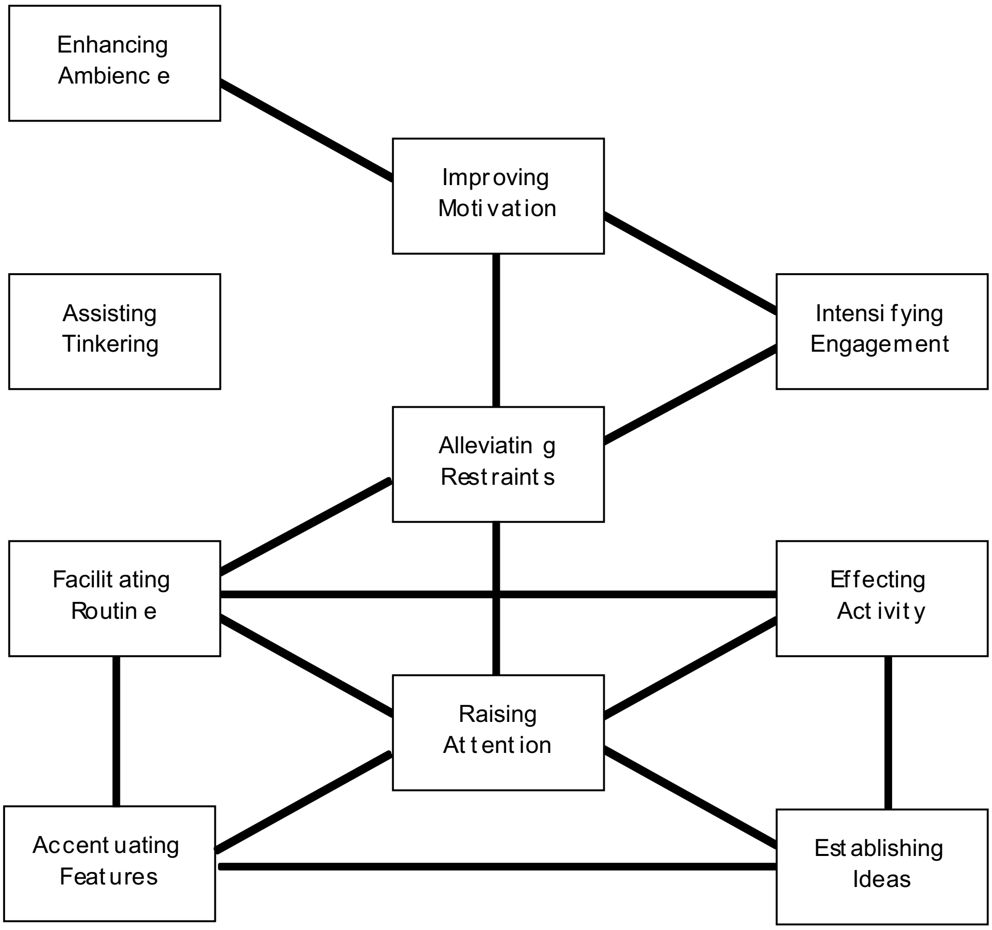
Indeed, contemporary theories of educational change, just like those of technological innovation, acknowledge how these processes are shaped by the sense-making of the agents involved (Spillane, Reiser & Reimer, 2002). Accordingly, conceptualisations of how teachers use curriculum materials have developed from rather limited views of teachers as simply following or subverting such materials, to more sophisticated perspectives encompassing teacher interpretation of, and participation with, curriculum materials (Remillard, 2005). Teachers necessarily incorporate the use of such materials into wider systems of classroom practice, so that the designs of curriculum developers turn out, in the words of Ball & Cohen (1996, p. 6), *"to be ingredients in –not determinants of– the actual curriculum"*. Hence, examining teacher response to new technologies, Kerr (1991; p. 121) has argued: *"If technology is to find a place in classroom practice it must be examined in the context of classroom life as teachers live it"*.

It is in this spirit, then, that I accept the invitation of Working Group 12 on Teaching Practices and Teacher Education to consider *"the role of the teacher… [in] the classroom use of technology"*, and to do so in a way which reflects the wider concern of that Group with *"ways of thinking about the classroom [which] try to capture… the essence of classroom activity"*. I will also accept, to a more limited degree, the invitation of Working Group 15 on Comparative Studies to look across countries and cultural settings at *"the role of the teacher"*, at *"teaching and learning materials (such as textbooks)"* and at *"the uses of technology [in] supporting educational practices"*.

2 A PRACTITIONER MODEL OF THE SUCCESSFUL USE OF NEW TECHNOLOGIES

In order to develop a better understanding of the appropriation of new technologies by classroom teachers, we undertook a study in Cambridge during the year 2000. This study investigated teachers' ideas about their own experience of successful classroom use of computer-based tools and resources (Ruthven & Hennessy, 2002). Teacher accounts were elicited through focus group interviews involving mathematics departments in secondary schools. These interviews were then analysed, qualitatively and quantitatively, so as to identify central themes and primary relationships between them.

At the left of the diagram are those themes most directly related to use of technology itself. It can serve as a means of: *Enhancing ambience* through changing the general form and feel of classroom activity; *Assisting tinkering* by helping to correct errors and experiment with possibilities; *Facilitating routine* by enabling subordinate tasks to be carried out easily, rapidly and reliably; and *Accentuating features* by providing vivid images and striking effects which highlight properties and relations. At the right of the diagram are those themes more directly related to major teaching goals. *Intensifying engagement* relates to securing the participation of students in classroom activity; *Effecting activity* relates to maintaining the pace and productivity of students during lessons; *Establishing ideas* relates to supporting the development of student understanding and capability. In an intermediate position lie the key bridging themes: *Improving motivation* through generating student enjoyment and interest, and building student confidence; *Alleviating restraints* through mitigating factors which inhibit student participation such as the laboriousness of tasks, the requirement for - and the demands imposed by- pencil-and-paper presentation, and vulnerability to mistakes being exposed; and *Raising attention* through creating the conditions for students to focus on overarching issues.



This model should not be read deterministically as implying that exploitation of the technological affordances on the left leads inevitably to achievement of the teaching aspirations on the right. Rather, each construct represents a desirable state of affairs which teachers seek to bring about in the classroom, to which they see the use of technology as capable of contributing. Of course, not all the components of the model were present in every example of successful practice, and some assumed more prominence than others. However, across the departmental interviews as a whole, all of the themes were invoked in the great majority of schools, indicating that they enjoy a wide currency. On the basis of this single study, the model had to be regarded as a tentative one. It was based only on teachers' decontextualised accounts of what they saw as successful practice, elicited through group interview, not on more strongly contextualised accounts of specific instances of practice, supported by examination of actual classroom events. Nevertheless, the model triangulated well against published case studies of technology use in ordinary classrooms, suggesting that it might be more widely transferable.

Subsequent studies have provided further validation of the model as an expression of teachers' ideals. One of these studies was carried out by the original team in Cambridge during 2004, but in a fresh group of schools which were professionally well regarded for their use of ICT in mathematics. Dynamic geometry, for example, was valued for *making student work with figures easier, faster and more accurate*, and consequently for *removing drawing demands which distract students from the key point* of a lesson. Various aspects of *making properties apprehensible to students through dynamic manipulation* were expressed. When a dynamic figure was dragged, students could *"see it changing"* and *"see what happens"*, so that properties *"become obvious"* and students *"see them immediately"*. The technology was seen as supporting teaching approaches based on *guiding students to discover properties for themselves.* What teachers suggested was that, while they might *"structure", "hint", "guide",* or *"steer"* students towards an intended mathematical conclusion, students could largely *"find out how it works without us telling them"* so that they were *"more or less discovering for themselves"* and could *"feel that they've got ownership of what's going on"* (Ruthven, Hennessy & Deaney, under review).

Another study which examined this model was carried out independently by members of the DIDIREM team in Paris during 2003 (Caliskan-Dedeoglu, 2006); an early report of its findings was given at CERME-4 (Lagrange, Dedeoglu & Erdogan, 2005). This examined the practice of experienced teachers, all longstanding classroom users of ICT, and involved in professional development networks. It, too, found that the model provided a useful template to describe teachers' pedagogical rationales for the classroom use of dynamic geometry. However, when teachers were followed into the classroom it became clear that these rationales sometimes proved difficult to realise in the lessons themselves. Teachers could be overly optimistic about the ease with which students would be able to use the software. Far from facilitating routine and alleviating restraints, the computer might then exacerbate them, with the teacher trying to retrieve the situation by acting primarily as a technical assistant. Equally, students could encounter difficulties in relating the figure on the computer screen to its paper-and-pencil counterpart. Rather than features being accentuated, they were called into question.

Similar problems were identified in the Cambridge study. But in both studies, some teachers were found to have developed strategies to address such problems. For example, we observed one teacher incorporate a lesson segment which served to bridge between dynamic geometry figures and their dissimilar pencil and paper counterparts. The problem of students having difficulties with the software was addressed in several ways. Bearing in mind the relative infrequency with which dynamic geometry was used in all of the classes concerned, such difficulties were sometimes pre-empted by the software being used only for teacher presentation. More commonly, use of the software was carefully structured, either by providing students with a prepared figure or by leading them through the construction process. Where students were expected to make fuller use of the software, one teacher encouraged them to employ a 'tidying' routine to eliminate the spurious points and lines that they often created onscreen as a result of their difficulties in physically manipulating the pointer (Ruthven, Hennessy & Deaney, 2005).

3 THE STRUCTURING CONTEXT OF CLASSROOM PRACTICE

What these two studies emphasise is that the model represents a guiding ideal for teachers. To be able to realise this ideal in practice, however, depends on teachers developing a craft knowledge to support their desired classroom use of new technologies. Those who have researched classroom teaching (Brown & McIntyre, 1993; Leinhardt, 1988) suggest that the complexity and significance of such knowledge is often overlooked. Craft knowledge is the largely reflex system of powerful heuristics which teachers bring to their classroom work. It draws on proceduralised and automated routines, tailored to the particular circumstances in which the teacher works. In particular, much proposed innovation entails modification of this system.

In this light, the following discussion will examine key components of the structuring context of classroom practice:

- working environment
- resource system
- activity format
- curriculum script
- time economy

In doing so, I will draw on a range of more general work on classroom processes and teaching craft, mindful of the observations of a number of studies of technology integration in mathematics lessons, particularly the Cambridge and Paris studies already mentioned, as well as Monaghan's work at Leeds (Monaghan, 2004), and the work of a multi-institutional French team involved in a national project on the use of CAS (Artigue, 2002; Guin, Ruthven & Trouche, 2005). I will also refer to several of the papers prepared for Working Groups at this conference.

3.1 Working environment

The use of ICT in teaching often involves changes in the working environment of lessons: change of room location, change of physical layout, change in class organisation, change in classroom procedures.

In many schools, ICT facilities are not available in ordinary classrooms, particularly at a sufficient level to allow students to work with them. Under these circumstances, lessons have to be relocated from the regular classroom to a computer laboratory. Such use of ICT facilities has to be anticipated by the teacher, and prevents more spontaneous and flexible use of the technology. It also entails disruption to normal working practices, and typically makes additional organisational demands on the teacher. Well-established management and support routines which help lessons to start, proceed and close in a timely and purposeful manner in the regular classroom (Leinhardt, Weidman & Hammond, 1987) may have to be adapted to suit the computer laboratory. Simply organising students to arrive at the computer laboratory rather than their regular classroom for the lesson, or moving them as a class between rooms during the lesson, introduces an extra demand.

Particularly when done on only an occasional basis, this also involves teacher and students adapting to an unfamiliar working environment. They are likely to be less familiar with the computer laboratory, and less likely to have customised its facilities. Moreover, the layout and facilities of such laboratories have often been designed for independent working, so that they are ill suited to public exchanges involving the whole class. This presses lessons towards activity formats based on students working independently under teacher supervision. Often, too, the number of workstations is smaller than the number of students in the class. Typically this leads to students being paired at a workstation. Alternatively, the class may be split in two, alternating between working at a computer and away from it. Neither of these is a common form of organisation in mathematics lessons in ordinary classrooms, and the second calls for careful management of transition procedures, and simultaneous supervision of two different activities.

As the provision of sets of handheld devices or laptop computers for use in ordinary classrooms becomes more common in schools, some of these issues are mitigated or removed. Nevertheless, in the ordinary classroom, machines typically have to be given out and set up, and later taken in and put away. Often, too, there is poor provision for students' work to be saved or printed, particularly in a form which can be integrated with their written work, and used in other lessons and at home. Likewise, the increasing availability of projection facilities and interactive whiteboards, both in ordinary classrooms and in computer laboratories, facilitates public exchanges involving the whole class. But this has also led to a quite different pattern of classroom computer use gaining currency in which teachers manage a single machine on behalf of the whole class.

One final issue is common to all these different working environments. Computer infrastructure in schools remains unreliable, leading not infrequently to lessons being disrupted. Indeed, teachers commonly report that they feel obliged to plan fallback lessons as a matter of course, against the ever-present possibility of technical difficulties.

While the modifications in working environment discussed here may bring ‘change’ and ‘variety’ to lessons, so ‘enhancing their ambiance’, they introduce new demands on teachers and students. Individually, each of these disruptions or additions to normal practice may amount to little. Cumulatively, however, they increase complexity and uncertainty, and call for significant adaptation of classroom routines.

3.2 Resource system

New technologies have broadened the types of resource available to support school mathematics. Educational suppliers now market textbook schemes alongside revision courseware, concrete apparatus alongside computer microworlds, manual instruments alongside digital tools. As many teachers realise, however, there is a great difference between a collection of resources and a coherent system.

Evaluations of the classroom use of computer-assisted instruction have frequently reported problems of mismatch between the CAI material and the regular curriculum (Amarel, 1983; Wood, 1998). These are exacerbated by a limited scope for teacher adaptation and reorganisation of CAI material. Such experiences have encouraged developers to offer greater flexibility to teachers. For example, the conference papers for Working Group 9 from Bueno-Ravel & Gueudet, and for Working Group 12 from Abboud-Blanchard, Cazes & Vandebrouck report on teachers’ use of a bank of e-exercises allowing them to design on-line worksheets for their students. These papers identify how teachers need to acquire the same depth of knowledge of the e-exercises as of textbook material in order to make effective use of them and to integrate them successfully with other classroom activity.

The printed textbook still remains at the heart of the resource system for studying mathematics in most classrooms. Textbooks are valued for establishing a complete, consistent and coherent framework, within which material is introduced in an organised and controlled way, appropriate to the intended audience. Increasingly, educational publishers are seeking to bundle digital materials with printed textbooks, often in the form of presentations and exercises linked to each section of the text, or applets providing demonstrations and interactivities. Such materials are attractive to many teachers because they promise a more straightforward and productive integration of old and new technologies than has been possible to date.

Textbook treatment of mathematical topics necessarily makes assumptions about what kinds of tools will be available in the classroom. Historically, these assumptions have been very modest. However, textbooks increasingly assume that some kind of calculator will be available to students. Well designed textbooks normally include sections which develop any techniques required for using these tools which cannot already be assumed, and establish some form of mathematical framing for them. However, it is rare to find textbooks taking account of other digital mathematical tools. Here, textbook developers face the same problems as classroom teachers. In the face of a proliferation in the tools available, which should be prioritised? Given the currently fragmentary knowledge about bringing these mathematical tools to bear on curricular topics, how can a coherent use and development be achieved?

Such issues are exacerbated when tools are imported into education from the commercial and technical world. Often, their intended functions, operating procedures, and representational conventions are not well matched to the needs of the school curriculum. Sometimes teachers get sucked into writing their own software templates much as they would paper worksheets, but often without the same ease; as described in the conference paper for Working Group 9 from Fugelstad. One solution is to design software specifically adapted to the needs of school mathematics. For example, the conference paper for Working Group 6 from Lagrange & Chiappini describes two pieces of educational software. One of these is designed for the teaching of functions and variation, and combines features of computer algebra and dynamic geometry. The other has been developed to provide versatile means of representing algebraic expressions and the solution sets of their associated equations and inequations. However, uncoordinated development of courseware resources devised with the treatment of particular mathematical topics in mind threatens a new proliferation of digital tools, and of their accompanying overheads of operational knowledge.

3.3 Activity format

Classroom activity is organised around formats for action and interaction which frame the contributions of teacher and students to particular lesson segments (Burns & Anderson, 1987; Burns & Lash, 1986). The crafting of lessons around familiar activity formats and their supporting classroom routines helps to make them flow smoothly in a focused, predictable and fluid way (Leinhardt, Weidman & Hammond, 1987). Indeed, this leads to the creation of prototypical activity structures or cycles for particular styles of lesson.

Monaghan (2004) worked with a number of secondary teachers who had made a commitment to move –during one school year– from making little use of ICT in their mathematics classes to making significant use. Over the course of the year he observed one 'non-technology' lesson taught by each teacher, as well as several 'technology' lessons, reporting that the two types of lesson tended to have quite different activity structures. In all the 'non-technology' lessons that he visited, a teacher-led exposition format was followed by an exercise-based seatwork format. Those 'technology' lessons which took place in the regular classroom using graphic calculators displayed this same structure. However, most of the 'technology' lessons involved students working with computers on 'open' tasks, often in the form of investigations. These lessons were organized so that a brief teacher-led introduction was followed by an extended period of worksheet-guided seatwork, monitored and assisted by the circulating teacher. As occurred with pre-computer, resource-based learning a generation ago, teachers spent more of the time in such lessons helping pupils to manage resources, and less helping them to reason mathematically.

Both the types of 'technology' lesson observed by Monaghan seem to have adapted an existing form of activity structure; either that of the exposition-and-practice lesson, or of the practical work and investigation lesson. Moreover, investigation lessons are typically seen as being appropriate only on an occasional basis in English schools, and Monaghan indicates that teachers saw use of technology as supporting such an approach. Under these circumstances, it is perhaps not surprising that no lessons of this type featured in Monaghan's small 'non-technological' sample. Likewise, in the studies we have carried out in Cambridge, many teachers have suggested that, by helping to create classroom conditions in which investigations can be conducted more successfully, particularly with lower attaining students, technology makes this activity structure a more viable option. While, in one sense, such use of technology is simply assisting teachers to realise an established form of practice, what is significant is that it may be enabling them to employ this practice more effectively and extensively.

However, many of the envisaged classroom uses of new technologies associate them with more radical change in activity formats, calling for new classroom routines. For example, Trouche (2005) proposes two pairwork activity formats, one around 'mirror observation' of problem solving activity, the other around use of a 'practical notebook' to report work on research tasks. In whole-class activity formats he introduces a 'sherpa student' responsible for managing the projected image of a calculator or computer screen. Each of these modifications of an established activity format calls for the establishment of new classroom norms for participation, and of classroom routines to support smooth functioning. For example, the conference paper for Working Group 9 from Bueno-Ravel & Gueudet discusses a range of activity formats through which teachers organised independent student work on e-exercises. Likewise, the conference paper for Working Group 9 from Miller, Averis & Glover characterises a system of whole-class activity formats for classrooms equipped with an interactive whiteboard, according to whether the teacher adopts the role of 'instructor', 'facilitator', or 'mediator'.

3.4 Curriculum script

In planning to teach a topic, and in conducting lessons on it, teachers draw on a matrix of professional knowledge. This knowledge has been gained in the course of their own experience of learning and teaching the topic, or gleaned from available curriculum materials. At the core of this matrix is a loosely ordered model of relevant goals and actions which serves to guide their teaching of the topic. This forms what has been termed a 'curriculum script' – where 'script' is used in the psychological sense of a form of event-structured cognitive organisation, which includes variant expectancies of a situation and alternative courses of action (Leinhardt, Putnam, Stein & Baxter, 1991). This script includes tasks to be undertaken, representations to be employed, activity formats to be used, and student difficulties to be anticipated.

For example, the conference paper for Working Group 12 from Abboud-Blanchard, Cazes & Vandebrouck, reports ways in which they saw teachers adapt their curriculum scripts to produce a better integration between e-exercises and other classroom work in the light of mathematical difficulties that students encountered. More fundamentally, the conference paper for Working Group 9 from Miller, Averis & Glover concludes that *"whilst it is relatively easy to draw the attention of teachers to the enhanced use of manipulations, to more developmental questioning, to the location of virtual manipulatives, and to encouraging interactivity, the fundamental need is for revised lesson planning [which] will involve new ways of thinking about lesson development"*.

In the studies which we have carried out in Cambridge, we have often found teachers viewing the use of new technologies through established curriculum scripts. For example, teachers frequently talk of how introducing the use of new technology has improved established practices, suggesting that it serves as a more convenient and efficient tool, or provides a more vivid and dynamic presentation. However, even at this level, new technologies may disrupt established curriculum sequences. One simple example comes from the work to establish a 'calculator aware number curriculum' for the primary school. Tackling realistic problems with a calculator leads to children encountering, and having to make sense of, negative and decimal numbers much earlier than envisaged in the established curriculum sequence. Equally, at secondary level, the ready availability of graphing technologies permits the use of more visual forms of argumentation within algebra, which calls for an earlier introduction of the mathematical underpinnings for such approaches. It is common for official curricula to recognise new tools such as these without anticipating implications for changed curriculum sequence, or for proliferating mathematical strategies. For example, the conference paper for Working Group 15 from Brown analyses the retrofitting of graphic calculators to examination curricula.

Some cultural features of curriculum scripts may influence the ease with which new technologies can be integrated. In particular, where there is a strong interest in mathematical anomaly and concern for mathematical rigour, issues of instrumentation may become more acute and demand more explicit attention. This helps to explain why the complexities of instrumentation have emerged much more strongly in European research than in North American. A North American researcher has suggested that European work has *"focused too narrowly on the applications of CAS to traditional algebraic symbol manipulation problems and ha[s] looked too hard to find subtle problems that are not well handled by CAS functions"*. He observes that *"learning how to use CAS functions to support applied problem solving is not as* *complicated or as fraught with the potential for mistakes as learning how to use the same tool for more general algebraic reasoning"* (Fey, 2006: p. 352).

3.5 Time economy

Time is a currency in which teachers calculate many of their decisions. It features strongly in the practitioner model of successful ICT use. The processes of 'facilitating routine' and 'raising attention' serve in 'effecting activity' in terms both of the pace of lessons and the productivity of students. Through 'accentuating features' and 'raising attention', the resulting 'time bonus' is converted into 'establishing ideas'.

In her study of dynamic geometry integration in the primary school classroom, Assude (2005) focuses on how teachers seek to improve the 'rate' at which the physical time available for classroom activity is converted into a 'didactic time' measured in terms of the advance of knowledge. Her study shows how important the refinement of new resource systems, activity structures and curriculum scripts is in improving this rate of didactic 'return' on time 'investment'. And there is a cost associated with innovation itself. While teachers are developing the requisite craft knowledge, lessons require more detailed preparation and are often conducted less efficiently and flexibly.

Old technologies typically remain in use alongside new, not just because they have a symbolic value, but because they make an epistemic contribution as much as a pragmatic one (Artigue, 2002). This 'double instrumentation' means that new technologies give rise to additional costs rather than to cost substitutions with respect to time. Teachers are cautious about new tools which require substantial investment, and alert for modes of use which reduce such investment and increase rates of return. Perhaps the most critical issue is the trade-off in terms of recognised mathematical learning between investment in students learning to make use of such tools against a safer conventional use of this time.

One of the teachers in our Cambridge studies actually used the language of cost-benefit analysis in drawing together the considerations behind his classroom mode of dynamic geometry use:

> *I don't get the students themselves to do their own constructions in dynamic geometry. I do them all for them… If I wanted the students to do it, it would take a long time in order for them to master the package and I think the cost-benefit doesn't pay there… And there's a huge scope for them making mistakes and errors, especially at this level of student… The students I was teaching, I think it would take me a long, long time to teach them the program and how to do it, and they would probably only end up with the same level of understanding as they've got at the moment, possibly a little bit more, but it just would be impractical… And the content of geometry at foundation and intermediate level just doesn't require that degree of investigation. So they need to learn certain facts, parallel line work, circle theorems, triangles, polygons, but most of those facts can be learned quite well enough without [dynamic geometry].*

Of course, as this example shows, such cost-benefit analysis depends on conceptions of how students' experience of the software might promote or inhibit mathematical learning. This relates to the conference paper for Working Group 9 from Assude on levels of instrumental and praxeological integration. In some classrooms in the Cambridge study, the development of new mathematical ideas was organised around teacher led presentation and questioning, with the teacher taking sole responsibility for use of the tool and exercising great care to avoid exposing apparent anomalies in its operation. Here the rationale was –in effect– to *minimise* instrumental demands on students; demands seen as unproductive, even obstructive, to mathematical learning. In other cases, the development of new mathematical ideas was organised around student activity framed and guided by the teacher, with students' use of the tool structured in ways considered beneficial for building mathematical knowledge – including having students construct figures for themselves, and having them encounter and make mathematical sense of apparent anomalies of operation. Here, the rationale was to *optimise* instrumental demands on students.

4 CONCLUDING THOUGHTS

We live at a time when there is an ever richer diversity of materials and tools available for use in the mathematics classroom. However, the main issue for curriculum developer and classroom teacher alike remains one of developing coherent use of a relatively small selection of them to form an effective resource system. This depends, in turn, on the more fundamental issue of coordinating working environment, resource system, activity format and curriculum script to underpin classroom practice which is viable within the time economy. This central challenge has tended to be overlooked in discussions of technology integration. It involves moving from idealised aspiration to effective realisation through the development of practical theory and craft knowledge.

What this implies for research in technology integration is the importance of taking a broad view which recognises that many of the same challenges confront other proposed innovations, and are, indeed, already encountered in existing practice. For example, there is scope for greater interaction to mutual advantage between what –within mathematics education– have been largely separate traditions of research on new educational technologies, on curriculum materials in use, and on pedagogical innovation. Likewise, it is also important for mathematics education as research field to encourage permeable boundaries and strong interactions with other areas of educational research. There is a great danger that research which focuses too exclusively on the distinctively mathematical dimensions of school practice may miss critical issues. The professional field of mathematics education is obliged to take a holistic view of such matters; to inform such a view, the research field needs to do likewise.

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DIGITAL TECHNOLOGIES: A WINDOW ON THEORETICAL ISSUES IN MATHEMATICS EDUCATION

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Abstract : In this lecture, I use digital technologies as a window on theoretical issues in mathematics education. More precisely I address the issue of the theoretical fragmentation of the field. After setting the problem in an introductory part, I successively use two dimensions of my research activity with digital technologies for approaching this issue. The first one corresponds to the development of the instrumental approach, the second one to the on-going research work on the integration of theoretical frames carried out within the European projects TELMA and ReMath.

I. Introduction

Researchers in the field of mathematics education seem more and more sensitive to the difficulties generated by the diversity and fragmentation of existing theoretical frames in the field[1]. Why so many theories and constructs? What exact needs do these constructions try to respond to? How many different constructs do we have for expressing more or less the same concerns, for fulfilling the same goals? What do they have in common and what exactly differentiates them? How do they respectively shape our research and design practices and more globally our vision of didactical action? How can we coherently and productively behave as researchers and as practitioners in such a world? Is it possible to reduce the increasing theoretical load of the field and, if so, how could this goal be better achieved? In the lecture, I address these issues, using as a filter the research developed on the design and educational use of digital technologies, and relying on my personal research experience. I specially use as examples the development of the so-called "Instrumental approach" from the mid nineties, and the research work I am currently involved in the frame of two European projects: TELMA and ReMath[2] with the aim of developing tools for connecting and integrating different theoretical approaches.

I.1. Why approaching theoretical issues through digital technologies?

I have chosen in this lecture to focus on theoretical issues in mathematics education, and to use research on teaching and learning with digital technologies: calculators, mathematics software or any kind of digital resources, as a window for looking at these issues.

This choice results from several reasons. First, there is no doubt that, more and more, our research community is sensitive to the diversity of existing theoretical frames in the field, and to the difficulties that result from it: difficulties in terms of appropriation of research 'problématiques' and results, in terms of exchanges and communication, in terms of capitalization of knowledge, in terms of relationships between research and practice. This

[1] Two recent issues of the Zentralblatt fur Didaktik der Mathematik (vol. 37.6 in 2005 and vol. 38.1 in 2006) have for instance focused on this issue, and this is also the theme of the contribution by Cobb to the new NCTM Handbook of Research on Mathematics Teaching and Learning (Cobb, 2007).

[2] TELMA : Technology Enhanced Learning in Mathematics is an ERT of the European Network of Excellence Kaleidoscope. ReMath: Representing Mathematics with Digital Media is a European IST (Project Number: IST4-26751).

sensitivity is attested by the increasing number of projects and publications addressing these issues[3], not to mention the existence at CERME4 (Dreyfus & al., 2006) and CERME5 (Bosch & al., this volume) of a specific working group devoted to these questions.

The second reason is linked to the fact that research involving digital technologies seems a good filter for investigating such issues. Evolution of research in that area indeed reflects the general trends and evolutions in mathematics education but it also presents some characteristics that make it a very interesting observatory when looking at the field. I will just briefly mention two of these characteristics.

- The first characteristic relates to the relationships between theory and practice. From the very beginning, digital technologies have been considered as a tool for educational change, a tool for forcing the implementation of didactic strategies more in phase with the theoretical frames and principles underlying didactical research, with the aim of course of improving teaching and learning, as evidenced for instance by the first ICMI Study on this theme (Cornu & Ralston, 1992). Unfortunately, the results are far from being those expected, making this area one of these where the tension between theory and practice is the most visible.
- The second characteristic relates to the specific dynamics of this area, which obliges research to face the proliferation of teaching and learning means resulting from a technological creativity and evolution whose constants of time are at odds with those of educational systems. This situation exacerbates instability and tensions, between educational systems and society, and inside the educational systems themselves.

Research on the educational use or development of digital technologies lives thus in a situation of permanent excitation and frustration which makes the questioning of the theoretical frames it relies on, the questioning of the links between theory and practice much more a necessity.

The third reason for the choice of technology is a more personal one. In the last ten years, my research has mainly focused on digital technologies, with an increasing sensitivity to theoretical issues. Due to different contractual projects I have been involved in, I have also become more and more sensitive to the links between theory and practice, and practice at different scales from the classical experimental project involving a few selected teachers and classrooms up to projects involving thousands of students.

As pointed out above, research dealing with digital technologies reflects the general trends and major evolutions of the field but it is also a source of inspiration for these. Along the last decades, the development of microworlds and educational involving them has thus been supported by constructivist approaches but it has also contributed to their progressive evolution (Papert, 1980), (Kafai & Resnick, 1996). The acknowledgement of the situated character of mathematical knowledge has reflected in the ideas of situated abstraction and webbing (Noss & Hoyles, 1996) but these ideas in turn have significantly contributed to the situated perspective in mathematics education. The increasing interest for the affordances of digital technologies in terms of representations have gone along with the increasing sensitivity paid to the semiotic dimension of mathematical knowledge in mathematics education and to the correlative importance given to the analysis of semiotic mediations but has also substantially contributed to it (Saenz-Ludlow & Presmeg, 2006), (Arzarello & Edwards,

[3] [3] Two recent issues of the Zentralblatt fur Didaktik der Mathematik (vol. 37.6 in 2005 and vol. 38.1 in 2006) have for instance focused on this issue, and this is also the theme of the contribution by Cobb to the new NCTM Handbook of Research on Mathematics Teaching and Learning (Cobb, 2007).

2005). Mechanical devices have played a major role in the diffusion of embodied perspectives (Nemirovsky R. & Borba, 2000), and now Internet technology plays a similar role as regards distributed cognition. Thus technology is without any doubt a reasonable window for looking at the existing theoretical diversity and at its implications, even if it gives to its study a specific flavour. It will be the filter used in this lecture for addressing the questions listed at the beginning of this introduction.

As announced above, for developing the reflection on these questions, I will rely on two examples, the first one being the development of what is often called now "the instrumental approach"[4], an approach I have contributed to in the last ten years. In coherence with the goals of this lecture, more than entering into the technical details of this approach, I find important to make clear why it emerged and what needs it aims at fulfilling. This is the reason why I will briefly enter in the story of its development.

II. The development of the instrumental approach

At the beginning of the nineties, I was asked by the Ministry of Education in France to join, as a didactician, a group of teachers, experts in the educational use of technology, working at identifying the potential offered by CAS for the teaching and learning of mathematics at secondary level, and at planning the curricular changes that the integration of CAS at senior high school level would require. This was a new kind of technology entering secondary education, much more powerful and complex than the graphical calculators mainly used at that time, and much more source of perturbation for the secondary mathematics education culture. And, not surprisingly, the first project which emerged from this collaboration raised a lot of questions.

This first research project carried out with DERIVE in computer labs showed an evident contrast between the idealistic discourse of the experts of the group, fully coherent with the literature on the educational use of CAS at that time, and what was revealed by the observations made in their classrooms (Artigue, 1997). It also allowed us to detect some possible sources for this contrast, and three of these especially attracted our attention:

- the conceptual/technical opposition permeating the literature and the experts' discourse,
- the poor sensitivity to the changes in the economy of mathematical practices induced by the use of CAS,
- the underestimation of instrumental issues.

A second project, bigger, involving several research teams[5], quickly followed, carried out with the first symbolic calculator: the TI92 just coming out. It allowed us to test our conjectures, deepen and structure our reflection, leading us to the development of the instrumental approach.

[4] I use here the article « the » for qualifying this approach even if it can appear inappropriate. Many approaches pay in mathematics education a great attention to instrumental issues, but this is certainly the one which is most focused on instrumental issues. The problem of connection between different instrumental perspectives will be considered in the third part of this text.

[5] This project involved beyond the DIDIREM team, a team in Rennes piloted by Jean-Baptiste Lagrange, in Montpellier piloted by Dominique Guin and Luc Trouche, in Grenoble piloted by Colette Laborde and in Lyon piloted by Gilles Aldon. I mainly collaborated with the two first ones.

II.1. The main choices underlying the instrumental approach

For developing this reflection, as explained in (Artigue, 2002), we had the feeling at that time that it was necessary to take some distance from the ordinary theoretical discourse in that area of research mainly inspired by constructivist perspectives. We needed a discourse able to consider concepts and techniques in more dialectical terms. We needed a discourse that would be less student centred, giving place to a wider systemic view on technological issues. We needed a discourse putting at the place it deserves the instrumental dimension of teaching and learning processes.

I am perfectly aware that these needs could have been fulfilled by different theoretical approaches, and through different constructs. But, to us, French researchers familiar with the Chevallard's anthropological approach (TAD in the following) (Chevallard, 1991, 1999, 2002, 2006) and also used to collaborate with researchers working in cognitive ergonomy, very soon imposed the idea that an appropriate combination of the TAD and of the views developed by Rabardel and Vérillon (Rabardel, 1995), (Vérillon & Rabardel, 1995) in cognitive ergonomy could provide the kind of thinking frame we were looking at. And this was how this instrumental approach emerged.

Said in a few words, didactical anthropology offered us the kind of macro-didactic framework we were looking at:

- it is institution centred and thus sensitive to institutional values and norms, and to the ways these shape teaching and learning processes;
- it envisions mathematics knowledge as something relative, emerging from mathematical practices, and is thus sensitive to everything affecting these practices;
- and not the less important, from its very beginning, it has developed a positive view about techniques, avoiding to reduce these to mere skills, making techniques a fundamental component of praxeologies, the construct used in this theory to approach human practices, showing up to what point techniques are conquests of the humankind.

Cognitive ergonomy for its part provided us with distinctions and constructs especially appropriate for looking at the role that digital tools can play in learning processes:

- first of all, through the fundamental distinction it makes between the inert object: the artefact and the instrument it can become for an individual, a community or an institution;
- second, by attracting our attention to the complexity of the instrumental genesis that produce this transformation from artefacts into instruments, and to the two intertwined dimensions of this process: the instrumentalisation directed towards the artefact and the instrumentation directed towards the subject;
- third, by obliging us to fully take into consideration that the tools of mathematical activity, whatever they be, fundamentally shape learning processes, their forms as well as their outcomes.

In our opinion, the TAD, even if it was sensitive to the role played by the diverse tools and ostensives supporting mathematical practices, had developed in a culture shaped by the traditional tools of educational practices in mathematics, and could not easily free itself from this cultural influence. For that reason, we considered very interesting to combine it with an approach such as that provided by Cognitive Ergonomy. Used to study learning processes at the workplace and human practices where digital technologies play a crucial role, it had developed another view of the changes induced by technological evolution in human practices and needs. But conversely, it did not seem enough sensitive to the legitimacy issues inherent to educational systems, and for instance to the fact that scientific and social legitimacy are not enough for ensuring didactical legitimacy. An appropriate combination of the two approaches seemed thus to us a priori more promising than the use of only one of these approaches, and this was the main reasons for our choice.

As was announced above, I will not enter here in technical details, but I would like to help those who are not familiar with this approach understand what changed for us thanks to this connection between two complementary perspectives, and why from that time we had the feeling that we were able to set up the questions related to technological integration in more appropriate terms, to better understand the observed limits of institutional efforts, and to think about what could provoke significant changes to the current situation.

II.2 The epistemic and pragmatic value of techniques

I will just take an example, that of the distinction that this approach led us to introduce between the epistemic and pragmatic value of techniques. Before entering in this example, I would like to point out that the formulation I have just used it itself a theoretical combination: techniques are, as mentioned above, fundamental objects in the TAD but the TAD does not distinguish between their epistemic and pragmatic values. These terms come from cognitive ergonomy. Nevertheless, in Rabardel's work they are attached to schemes not to techniques. Schemes of instrumented action have a pragmatic value because they produce results, they make us able to transform the world around us, but they have also an epistemic value because they contribute to our understanding of the objects they involve.

This distinction, applied to techniques, acted for me as a catalyst and I could no longer see the question posed by the integration of digital technologies as I did before. The resistance to technology, the incredible recurrence of debates on topics such as the long division, could be now re-interpreted in terms of problematic balance between the epistemic and pragmatic value of instrumented techniques, and a new vision emerged from that. I try to synthesize it in the figure 1 below and in the comments accompanying it.

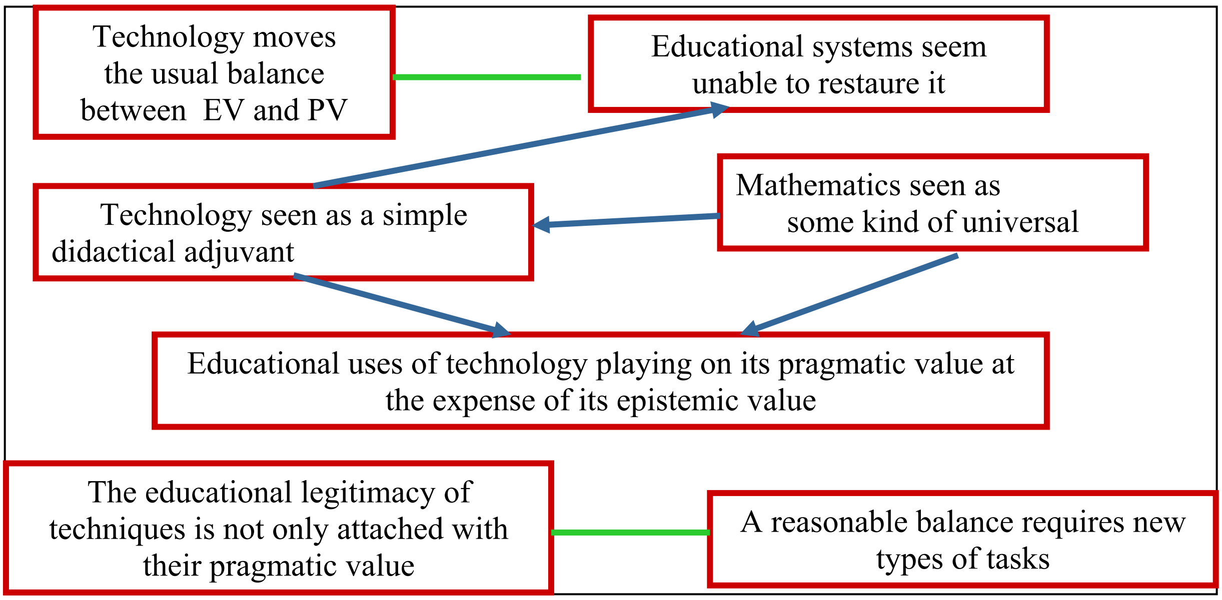

Figure 1: How digital technologies affect the usual balance between the epistemic and pragmatic value of techniques

Digital technologies boil over the traditional balance in education between the epistemic and pragmatic values of techniques, a balance historically built and established in a paper and pencil culture (even if calculations have always been instrumented by abacus, numerical tables, mechanical devices...). Educational systems meet evident difficulties at reacting appropriately to this change. But these difficulties are not independent on the way educational systems tend to adapt to technology, giving it no more than a role of pedagogical adjuvant. Roughly speaking, technology must help students learn quicker, easier and better more or less the same mathematics. This position is itself linked to a vision of mathematics as a field of knowledge universal by essence both in time and space, culturally dominant. I claim, and of course have arguments for supporting this claim even if I do not develop them here, that such a thinking frame generates an imbalance, inducing an educational use of technologies playing on their pragmatic power at the expense of their epistemic power. Ken Ruthven in his plenary lecture at CERME5 has expressed similar views even if he has used another terminology (Ruthven, this volume). But what makes legitimate a technique in an educational system cannot only be its pragmatic power. This makes an essential difference between School and the outside world.

Making technology legitimate and mathematically useful from an educational point of view, whatever be the technology at stake, requires modes of integration that provide a reasonable balance between the pragmatic and the epistemic values of instrumented techniques. This, as shown by research, if one correctly reads its results, requires tasks and situations that are not simple adaptations of paper and pencil tasks. It requires situations that very often cannot be thought of in a paper and pencil environment. One can easily imagine that such tasks are not those that a teacher designs first when entering in the technological world with a paper and pencil culture, which has been and is still the general case. From this point of view, the research carried out in Grenoble with Cabri-géomètre studying the evolution over three years of the scenarios built by a group of teachers having differentiated relationships with technology is especially illustrative (Laborde, 2001).

Even if my argumentation has been schematic, I hope that the reader will understand how thinking in such terms changes one’s mind, obliges to look at resistances differently, obliges also to question the resources, the support that, as researchers, we provide to teachers and institutions for overcoming the difficulties they unavoidably meet when trying to integrate technological tools.

I have taken just an example: a distinction that a particular theoretical construction led us to introduce; it could be seen as a pure anecdote. For me, it is much more than that. The interest and importance of a theoretical construction is intimately attached to the way it obliges us to move our ordinary lens, making visible phenomena invisible before, helping us question our “spontaneous” interpreting frames, making rational and understandable behaviours apparently erratic or incoherent, in a word changing our vision and showing us where we have to put our intelligence and energy. For me, regarding technological issues, the instrumental approach has played such a role.

II.3. Further developments of the instrumental approach

In the last ten years, more and more researchers have fruitfully explored the potential of this combination between TAD and cognitive ergonomy for looking at the integration into mathematics education of different technological tools, CAS of course (Guin, Ruthven & Trouche, 2004), (Lagrange, 2005) but also spreadsheet (Haspekian, 2005a, 2005b), (Haspekian & Artigue, 2007) and dynamic geometry software (Laborde & al., 2006). The first constructs have thus been tested and consolidated in some kind of continuous progression.

Today, we have entered another phase due to the use of this approach for dealing with a new category of resources: on line and tutorial resources, on the one hand and to the move in research focus from the student to the teacher on the other hand (Monaghan, 2004). In my research team, DIDIREM, this is taking place in the frame of two different projects. The first is a national project piloted by Lagrange - the GUPTEN project (Abboud-Blanchard, Lagrange, 2006) – focusing on the understanding of teaching practices in digital technology environment. The second is a regional project involving more than 5000 grade 10 students, resulting from the fact that three years ago the Region Ile-de-France decided to support high school students living in poor social areas by paying them access to on-line mathematics tutorial resources. Through this action carried out in collaboration with the academic structures, the Region tried to compensate the limited access that these students have to the many services for accompanying personal school work existing in France, for financial reasons[6].

These changes in the technology used and in focus today challenge the instrumental approach and questions such as the followings enter the scene:

- What does it mean for a student to transform such tutorial resources into a learning instrument, and what kind of instrument do they become?
- What does it mean for a teacher to turn them into a professional instrument and what results from it?
- What do these instrumental genesis have in common with those we have been used to study in the last ten years? What is different? And why? What didactical phenomena result from these differences? How to manage these?

The results we have obtained in the recent years are still a bit fragmentary but patterns begin to emerge, partly different from those we are more familiar attached to the instrumentation of open technologies such as CAS, spreadsheets, dynamic geometry software or graphic calculators (Abboud-Blanchard, Cazes & Vandebrouck, this volume), (Artigue & al., in press), (Vandebrouck, in press). They are different both on the side of the students and on the side of the teachers, and these differences themselves are insightful, both from a theoretical and a practical point of view. Up to now, the main focus of attention in the instrumental approach has concerned mathematical objects and the ostensive attached to them. What was at stake was the understanding of the productive and problematic dimensions of the computer transposition of mathematics knowledge achieved by the artefact (Balacheff, 1994) and of its consequences both from an institutional and personal point of view. With tutorial programs, these effects are not necessarily the most visible when one tries to interpret students' behaviour in terms of instrumental genesis. They are partly hidden by other instrumental processes seemingly more tactical, as if students tried to discover and adapt to the rules of a new didactic contract. An evident reason for that is that these programs implement mathematical and didactical praxeologies, and that the characteristics of the interaction with mathematical knowledge they propose often make a didactical adaptation more likely to ensure success than an adidactical one (Brousseau, 1997). On the teacher side, for the same reasons, instrumental genesis is also very different from those already observed.

II.4 Tensions between schemes and techniques

Before ending with the example of the instrumental approach, I would like to point out another facet of this story. I have perhaps given the false impression that combining the TAD and cognitive ergonomy was a productive and non problematic task. This is not exactly the

[6] All documents attached to this project are accessible on the website of the IREM Paris 7: http://pcbdirem.math.jussieu.fr/SITEscore/accueil.htm

case. It was certainly productive but the construction was not exempt of tensions. The two theoretical perspectives combined in it obey indeed quite different coherences. For the first perspective, basic objects are institutions. In it, individuals exist through the different institutional subjections they are submitted to, they emerge from these. The second perspective is a socio-cultural approach, inspired by Activity Theory but also by the theory of conceptual fields (Vergnaud, 1990), and is clearly a cognitive theory. This difference reflects in the evident tension existing between on the one hand the language of praxeologies and techniques used in the TAD, and on the other hand the language of schemes used by Rabardel. This tension between schemes and techniques, reflecting a fundamental tension between the institutional and the individual has been extensively discussed in the recent years as attested by the proceedings of the recent CAME Conferences but up to now has not been overcome. Each of us manages it in some sense in a particular way according to his particular culture and sensitivity, according to the particular context and aim of his/her research. For me, this is a good illustration of the difficulties that one necessarily meets when trying to integrate two different logics, to build something starting from two different coherences. It shows the difficulties raised by the connection of theoretical frames, and contributes to explain why, so many often, we prefer to develop ad hoc and local constructions.

This connection and integration of theoretical frames is nevertheless something I have been involved in the recent years in the frame of two European projects dealing with technology: the TELMA and the ReMath projects[7]. And I will use these two projects for progressing in the reflection.

III. Connecting and integrating theoretical frames: TELMA and ReMath[8]

TELMA (Technology Enhanced Learning in Mathematics) is a European Research Team, a substructure of the European Network of Excellence Kaleidoscope created in 2003, which has among its main aims to provide tools for improving the exchange and mutualisation of knowledge between teams working in technology enhanced learning, and make them able to propose and carry out joint projects. The research team TELMA, involving six teams from four different countries: Italie, United Kindom, Greece and France, focuses on mathematics. One hypothesis made in TELMA is that the multiplicity and isolated character of most theoretical frames used in technology enhanced learning in mathematics is an obstacle to the exchange and mutualisation of knowledge, and that the development of collaborative work requires better mutual understanding of our respective theoretical frames.

For doing so, in TELMA we first collectively worked on a selection of published papers prepared by each team. This first study confirmed the existing theoretical diversity among us, allowed a first synthetic description but let us unsatisfied at least for the following reason. In the selected research papers, theoretical references were of course explicit but it was very difficult to infer from what was written the exact role theoretical frames had played in the design of the research project, in its management, in the analysis and interpretation of the collected data, and in the results finally obtained. The ways theoretical frames were dealt with, mainly by invocation, visibly obeyed some kind of didactic contract proper to research publications but was not really insightful. We thus felt that it was time to move from the declarative to the operative, and to seriously investigate the exact role that theoretical frames played in the design and analysis of educational uses of technology in mathematics (our

[7] TELMA (respectively ReMath) deliverables and documents are accessible on the website (respectively).
[8] Publications attached to these projects are accessible on line : http://telma.noe-kaleidoscope.org and http://www.remath.cti.gr

common interest), in order to clarify what was useful and possible to aim at in terms of theoretical integration or networking between us, and how this could be reasonably achieved.

III.1. TELMA methodological tools

For that purpose, we progressively developed some methodological tools, among which an original strategy of cross-experimentation (Cerulli & al., this volume). Each team was asked to prepare and carry out an experimentation involving the use of an alien technology, that is to say of a technological tool designed by another team from another country. Of course, each experimentation had its specific goals but each was also by itself an object of collective research for our group. Guidelines were collectively established for monitoring the whole process from the design and a priori analysis of it to its implementation, collection of data and a posteriori analysis. This was achieved by the young researchers of each team who were in charge of the experimentation, another methodological decision whose rationale will be clarified below. And beyond that, reflective interviews based on stimulated recall were a posteriori organized with these young researchers in order to make clear the exact role theoretical frames had played in the different phases of their experimental work, explicitly or in a more naturalized way.

Why such a methodology? Roughly speaking, for making visible the invisible, explicit the implicit. We thus hypothesized that introducing an alien technology would make problematic and thus visible design decisions and practices that generally remain implicit in the research teams when these use tools built within their research and educational culture, and that this visibility would be increased by the guideline requirements. The delegation to the collective of the young researchers of the organization of the cross-experiment was also in our opinion a factor contributing to this visibility, by the distance created with more standard experiments.

Beyond that, the methodology of cross experimentation made also possible the comparison of the designs and analysis produced for the experiments with those already produced by the teams having designed and produced the tools. Moreover, most tools being experimented by two different teams, it was also possible to compare their designs, implementations and analysis. All these comparisons were expected to contribute to the visibility of the role played by theoretical frames and contexts, and help understand their respective influence.

For supporting the organization of this cross-experimentation and allowing its exploitation according to our aims, we needed a common language independent of our respective theoretical frames, and we built it around the notion of *didactical functionality* and the idea of *concern*. The first notion comes from the hypothesis that behind any educational use of a technological tool lies, more or less explicit, the identification of some didactical functionality for this tool, and that this identification is influenced by the theoretical frames and principles one relies on. Through this notion, in fact, our plan was to approach theories through their impact on design and use, and the guidelines for the cross-experimentation tried to make the choices made in terms of didactical functionalities as explicit as possible.

The notion of didactical functionality was defined as a three dimensional object, these dimensions being not independent (Cerulli & al., 2006). A particular didactical functionality attached to a tool is thus defined by:

- a set of characteristics of the tool,
- a specific educational goal exploiting these characteristics,
- and a set of modalities of employing the tool in a teaching/learning process referred to the chosen educational goal.

The second basic notion is that of concern. Concerns situate as a rather general level: cognitive, epistemological, social… Behind this choice, is the hypothesis that the level of concerns is a good level for establishing connections between theoretical frameworks, as it allows researchers to focus on the functionality of these frameworks and on the needs they respond to. As explained in the first TELMA deliverable on theoretical frames (Artigue, 2005), concerns are what we can most easily share and agree on, but:

> "In different frameworks, not all of these concerns are considered or given the same emphasis, and even when they seem to be given a similar importance, they are not necessarily expressed, dealt with in the same way, with the same conceptual tools, and the decisions taken can diverge. This is through the identification of the respective attention given to these different concerns, and the precise ways they are approached that we try to elucidate the role played both explicitly and implicitly by theoretical frameworks, to identify interesting connections and complementarities, and also divergences, potential misunderstandings and conflicts we need to be aware of." (p. 44)

A set of concerns was thus a priori attached to each of the dimensions of didactical functionality, and I list below those attached to the first dimension: that of tool analysis. They result from the analysis carried out during the first phase of the project and are the following:

- concerns about tool ergonomy,
- concerns about the implementation of mathematical objects and of the relationships between these objects,
- concerns about possible actions on objects,
- concerns about semiotic representations,
- concerns about interaction between students and mathematical knowledge,
- concerns about interaction with other agents (real or virtual)
- concerns about the support provided to the professional work of the teacher
- concerns about institutional and/or cultural distances

III.2 Some lessons from the cross-experimentation

This was a fascinating experience. As expected, the perturbation created by the introduction of alien technologies and the methodological choices, made it quite insightful, giving us a vision of the role that theoretical frames play in the design and analysis of educational uses of digital technologies quite different from what can be inferred from the literature, and, as a consequence, influencing our vision of integration or networking needs and possibilities (Cerulli, 2007), (Artigue, 2007).

The first evidence provided by the cross-experimentation was that theoretical frameworks, while having influenced design and analysis, were far from having played the role they are usually given in the literature. They mainly acted in the design as implicit and naturalized frames, and more in terms of general principles than of operational constructs. Even if some interesting variations can be noticed, all the teams pointed out the gap they experienced between the support offered by theoretical frames and the decisions to be taken in the design process. A good deal of the design work could be better labelled as artcraft work than engineering work.

This does not mean that theoretical frames did not have a serious influence on the identification of didactical functionalities and thus on the design. For instance, the influence of the theory of didactical situations and of the TAD was evident in the choices made by the French teams. It was clear that they were expecting from the tools they would use to provide a "milieu" for the students' work with a strong potential in terms of a-didactic adaptation. This led them to pay particular attention to the feedback that the artefact could offer to students' actions. They were also very sensitive to the necessity of maintaining a reasonable distance between the mathematics implemented in the artefact and the French institutional ones, and to limit the instrumental needs, this sensitivity being increased in that particular case by the reduced duration of the experiment (about one month). This influenced in an evident way the selection of the artefacts, the specific educational goals attached to them and the scenarios built. The other teams did not seem to have the same sensitivities and thus did not impose to their constructions the same constraints. They were more open to exploration activities, trying to benefit from the potential that the selected artefact offered in terms of semiotic mediation for instance, they did not feel so obliged as the French researchers to anticipate the possible mathematical outcomes of the student's interaction with the artefact, the possible sharing of responsibilities between the students and the teacher in order to optimize it, to think about what could be institutionalized and how from the students' activity. This resulted in very different constructions, and constructions quite different from those that the developers of the artefacts could have made. For instance, an Italian team working with the French software Aplusix invented a very innovative use of this software, quite different from all those already observed in France, for being coherent with its own culture.

I will not enter into more details but I would like to stress that the differences we observed were not just resulting from differences in theoretical approaches – the two Italian teams sharing the same didactic culture and using the same artefact built rather different designs. What we see here is more an intertwined influence of theoretical and contextual constraints. For instance, it was evident that the institutional pressure was stronger in France than in Italy and Greece, reducing the space of freedom for organizing such an experimentation. And retrospectively we can think that this is not by chance that the TAD for which institutions are the basic objects developed first in France. Theories are also cultural objects.

Recently, the TELMA project has given birth to another European project called ReMath focusing more specifically on representations but ambitioning to extend the networking effort towards the design of digital artefacts itself. This is for us a real new challenge, the theoretical diversity being increased by the necessity to incorporate those used by the neighbouring community of AIED (Grandbastien & Labat, 2006), but we engage in it with a clearer vision of what we need and what we can reasonably achieve in terms of theoretical integration (Artigue, 2006), and this leads me to the conclusion of this lecture.

IV. Conclusion

In the last ten years, I have the feeling that I have learnt a few things about theoretical diversity, and about the ways of dealing with it. I will try to summarize these below.
The first is that theoretical diversity is something inherent to our field. The educational field we work in, even restricted to mathematics, has so many different facets, is so dependent on contexts and cultures that theoretical diversity, even if it has to be controlled, imposes as an evidence.
The second is that building or choosing some theoretical approach is choosing some coherent lens for looking at the "real world" we want to understand and improve, and that our theories are powerful because they renounce to be holistic.

The efforts we have to make for improving communication and capitalization of knowledge will be counter-productive if they do not respect the existing coherences or misunderstand the real nature of these. For that reason, I am deeply convinced that the unification metaphor is not an appropriate metaphor and that more than integration we have to look for networking.

What can be the ways towards such a networking, what makes it possible? The different groups and individuals that constitute our community have different sensitivities shaped by the social, cultural and educational contexts they live in, by their particular history. But, in spite of such cultural differences, they face also common problems, educational dynamics and phenomena which are not so different; they are subjected to similar global influences. They share thus some common concerns, even if they approach these differently, building on the perspectives and constructs they are familiar with.

Networking theoretical frames can be achieved in my opinion by building a meta-coherence on these common concerns and needs, trying to understand why we respond to these as we do, and trying to benefit from the creativity of our different answers without loosing our identity. This is not an easy task. It cannot be achieved by a superficial look at our respective constructions. It cannot be achieved just by reading and listening. Knowledge in this domain as in any other can only result from collective practice organizing in appropriate ways the meeting of different cultures.

My experience leads me to think that this is not a utopia, and that boundary objects, with the meaning of this term in communities of practice (Wenger, 1998) can support the communication. Preparing this lecture, I was wondering what kinds of objects have more or less played this role in the TELMA work. I would say that the notion of instrument was certainly for us such an object, present in our different didactic cultures and responding to close concerns. Another one, not expected, has been perhaps the notion of a priori analysis, which has become progressively shared, not of course for each of us with the meaning given to it in the theory of didactic situations (Brousseau, 1997) where it originated but filled with what our different approaches found reasonable to try to anticipate and control. Through such boundary objects, relying on the concerns we share even if we more or less focus on these in our respective approaches, we can certainly increase our mutual understanding, organizing the diversity we face, and turning into a force of the field what we perceive today as an obstacle. This can be neither the work of an individual researcher, nor of a particular team, but this can certainly be a collective ambition, and an ambition for an association such as ERME.

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